Information in Black Hole Radiation

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Abstract

In this report we discuss the fidelity and the rate at which we can retrieve the information that was thrown into a black hole. The analysis is based on the paper by Preskill and Hayden [1].

1 Introduction

The question we look into is that if Alice throws a k-qubit quantum information M into a black hole H, can Bob tell what the information was by looking at the Hawking radiation, given that Bob holds a quantum memory that is perfectly correlated with the previously emitted Black Hole radiation E. We assume the black hole dynamics to be unitary and rapidly mixing. Also, consider a reference system N which is initially maximally entangled with M. By retrieving information of M from the black hole, we mean extracting a subsystem of size $|M|$ which is maximally entangled with N, so that Bob can do anything with it that he would have been able to do with M before Alice threw it into the black hole.

Just after Alice throws the information into the black hole, the system B ($= HM$) is transformed by a unitary transformation V^B chosen uniformly with respect to a Haar measure. After some time, the black hole is in state B' and has radiated system R which Bob observes.

Remark (Haar measure)**.** *A complex vector in $\mathbb{C}^n$ may be represented using a vector in $\mathbb{R}^{2n}$. Unitary transformations preserve the length of this vector and hence keep it within an S^{2n-1} subspace of R^{2n}. Haar measure corresponds to the measure being rotationally invariant on this sphere. The expected density matrix of pure state chosen uniformly using a Haar measure is a maximally mixed state.*

2 Entanglement in Black Holes

Consider first the entanglement between H and E [2]. When the black hole is just formed, $|E| << |H|$ and the radiation E is maximally entangled with the black hole state H. As the evaporation proceeds, $\log |H|$ declines to half it's value and soon after $|H| << |E|$. The black hole state H is then maximally entangled with radiation E.

We now give a proof of the above [3]. Consider a bipartite pure state $|\psi\rangle$ composed of subsystems A and B. The state is chosen uniformly with respect to a Haar measure. Consider the flip operator $\mathbb{F}$: $\mathbb{F}(|\varphi_i\rangle \otimes |\varphi_j\rangle\ ) = |\varphi_j\rangle \otimes |\varphi_i\rangle$.

$$\begin{aligned}\mathrm{tr}((\rho\otimes\rho)\mathbb{F}) &= \sum_{i,j=0}^{n} \langle\varphi_i\varphi_j|(\rho\otimes\rho)\mathbb{F}|\varphi_i\varphi_j\rangle = \sum_{i,j=0}^{n} \langle\varphi_i\varphi_j|(\rho\otimes\rho)|\varphi_j\varphi_i\rangle \\ &= \sum_{i,j=0}^{n} \langle\varphi_i|\rho|\varphi_j\rangle\langle\varphi_j|\rho|\varphi_i\rangle = \sum_{i=0}^{n}\langle\varphi_i|\rho^2|\varphi_i\rangle = \mathrm{tr}(\rho^2)\end{aligned}$$

We use this by creating a copy $A' \otimes B'$ of the system $A \otimes B$.

$$\begin{aligned}\mathbb{E}_\psi \mathrm{tr}(\rho_A^2) &= \mathbb{E}_\psi \mathrm{tr}[(\rho_A\otimes\rho_{A'})\mathbb{F}_{AA'}] = \mathbb{E}_\psi \mathrm{tr}[(\psi_{AB}\otimes\psi_{A'B'})(\mathbb{F}_{AA'}\otimes\mathbf{1}_{BB'})] \\ &= \mathrm{tr}\left[(\mathbb{F}_{AA'}\otimes\mathbf{1}_{BB'})\int|\psi\rangle\langle\psi|\otimes|\psi\rangle\langle\psi|d\psi\right]\end{aligned}$$

We now compute

$$Z = \int |\psi\rangle\langle\psi| \otimes |\psi\rangle\langle\psi| d\psi = \int (U \otimes U)(|\psi_0\rangle\langle\psi_0| \otimes |\psi_0\rangle\langle\psi_0|)(U^\dagger \otimes U^\dagger) dU = \int (U \otimes U)\rho_0 (U^\dagger \otimes U^\dagger) dU$$

where U are unitary matrices. For any unitary matrix V, we have

$$(V \otimes V)Z = \int_U ((V \otimes V)(U \otimes U)\rho_0 (U^\dagger \otimes U^\dagger) dU = \int_W (W \otimes W)\rho_0 (W^\dagger \otimes W^\dagger)(V \otimes V) dW = Z(V \otimes V).$$

Therefore Z commutes with every unitary matrix $U \in U(n)$ representation. By Schur's lemma, Z can be decomposed as a sum of projection operators on the invariant subspaces of $\mathbb{C}^n \otimes \mathbb{C}^n$.

$$Z = \lambda_1 \pi_{\text{sym}} + \lambda_2 \pi_{\text{antisym}} \qquad \pi_{\text{sym}} = \frac{1}{2}(\mathbf{1}_{n^2} + \mathbb{F}) \text{ , } \pi_{\text{antisym}} = \frac{1}{2}(\mathbf{1}_{n^2} - \mathbb{F}).$$

Noting that $\mathbb{F}$ commutes with all $U \otimes U$

$$\text{tr}(\mathbb{F}Z) = \int \text{tr}(U \otimes U(\pi_{\text{sym}} - \pi_{\text{anitsym}})\rho_0 U^\dagger \otimes U^\dagger dU = \text{tr}(\rho_0 \pi_{\text{sym}}) - \text{tr}(\rho_0 \pi_{\text{antisym}}) = \lambda_1 \text{tr}(\pi_{\text{sym}}) - \lambda_2 \text{tr}(\pi_{\text{anisym}})$$

$$\text{tr}(\mathbf{1}_{n^2} Z) = \int \text{tr}(U \otimes U(\pi_{\text{sym}} + \pi_{\text{anitsym}})\rho_0 U^\dagger \otimes U^\dagger dU = \text{tr}(\rho_0 \pi_{\text{sym}}) + \text{tr}(\rho_0 \pi_{\text{antisym}}) = \lambda_1 \text{tr}(\pi_{\text{sym}}) + \lambda_2 \text{tr}(\pi_{\text{anisym}})$$

and hence, using $\rho_0 = |\psi_0\rangle\langle\psi_0| \otimes |\psi_0\rangle\langle\psi_0|$,

$$\lambda_1 = \frac{2}{n(n+1)} \text{tr}(\pi_{\text{sym}} \rho_0) = \frac{2}{n(n+1)} \qquad \lambda_2 = \frac{2}{n(n-1)} \text{tr}(\pi_{\text{antisym}} \rho_0) = 0.$$

Therefore,

$$\begin{aligned}
\mathbb{E}_\psi \text{tr}(\rho_A^2) &= \text{tr}\left[(\mathbb{F}_{AA'} \otimes \mathbf{1}_{BB'})\left(\frac{2}{n(n+1)} \pi_{\text{sym}}^{AB:A'B'}\right)\right] = \frac{2}{n(n+1)} \text{tr}\left[(\mathbb{F}_{AA'} \otimes \mathbf{1}_{BB'})\frac{1}{2}(\mathbf{1}_{ABA'B'} + \mathbb{F}_{ABA'B'})\right] \\
&= \frac{1}{n(n+1)} \text{tr}\left[(\mathbb{F}_{AA'} \otimes \mathbf{1}_{BB'}) + (\mathbf{1}_{AA'} \otimes \mathbb{F}_{BB'})\right] \\
&= \frac{|A||B|^2 + |A|^2|B|}{|A||B|(|A||B|+1)} = \frac{|A| + |B|}{|A||B| + 1}
\end{aligned}$$

For $A >> B$, we have $\mathbb{E}_\psi \text{tr}(\rho_A^2) \approx 1/|A|$. Let a_i be the eigenvalues of ρ_A, then

$$\sum_i \left(a_i - \frac{1}{|A|}\right)^2 = \left(\sum_i a_i^2 - \frac{2}{|A|}\sum_i a_i + \frac{|A|}{|A|^2}\right) = \text{tr}(\rho_A^2) - \frac{1}{|A|} \approx 0$$

Therefore, A is almost maximally mixed. Using Levy's Lemma we can say that for almost all pure states $|\psi\rangle$, ρ_A is close to maximally mixed [4].

Theorem (Levy's Lemma)**.** *Let $\phi : S^{2n-1} \to \mathbb{R}$ be a Lipschtz continuous function on the unit sphere, i.e. $|\phi(\mathbf{x}) - \phi(\mathbf{y})| \leq \eta \|\mathbf{x} - \mathbf{y}\|_2$. Then*

$$\textit{Prob}\Big[|\phi(\mathbf{x}) - \mathbb{E}_{\mathbf{x}}\phi| \geq \epsilon\Big] \leq 2\exp\left(\frac{-n\epsilon^2}{9\pi^3\eta^2}\right)$$

The local trace distance $\phi : |\psi\rangle \mapsto \|\rho_A - \frac{\mathbf{1}_A}{|A|}\|_1$ is Lipschitz continuous.

3 Fidelity of Retrieved Information

Consider the case when $|H| << |E|$ so that H is maximally entangled with E. Just after Alice throws in her information, the black hole system B is maximally entangled with NE. As the information leaks from the black hole through the radiation R, the correlation between N and remaining black hole state B' weakens, and the information in M goes to Bob. Suppose s qubits have radiated in R, and $n-s$ qubits remain in B'.

Let $\Psi^{\rm BNE}$ be the pure density operator of BNE, and $\rho^{\rm BN} = {\rm Tr}_E \Psi^{\rm BNE}$ be the corresponding marginal density operator on BN. The marginal density operator on NB' is given by

$$\sigma^{\rm NB'}(V^B) = {\rm Tr}_R\Big[\rho^{\rm NB}(V^B)\Big] \qquad , \text{ where } \rho^{\rm NB}(V^B) = \Big(I^N \otimes V * B\Big)\rho^{\rm NB}\Big(I^N \otimes V^{B\dagger}\Big).$$

We show that this marginal density operator is very close to a maximally mixed and B' and N are almost separable. To this end we prove the following inequality

$$\int dV^B\ \|\sigma^{\rm NB'}(V^B) - \sigma^N(V^B) \otimes \sigma^{B'}_{\max}\|_1^2 \le \frac{|NB|}{|R|^2}{\rm Tr}\Big[\big(\rho^{\rm NB}\big)^2\Big]$$

where $\sigma^N(V^B) = {\rm Tr}_{B'}\big[\sigma^{\rm NB'}(V^B)\big]$ is the marginal density operator on N, and $\sigma^{B'}_{\max} = I^{B'}/|B'|$ is the maximally mixed density operator on B'.

$$\|\sigma^{\rm NB'}(V^B) - \sigma^N(V^B) \otimes \sigma^{B'}_{\max}\|_2^2 = {\rm Tr}[(\sigma^{\rm NB'}(V^B))^2] - \frac{1}{|B'|}{\rm Tr}[(\sigma^N(V^B))^2].$$

As was done in the previous section, and using the cyclic property of the trace

$$\begin{aligned}
\int dV^B\ {\rm Tr}[(\sigma^{\rm NB'}(V^B))^2] &= \int dV^B\ {\rm Tr}[(\sigma^{NB}(V^B) \otimes \sigma^{\bar N\bar B}(V^{\bar B}))(F^{N\bar N} \otimes F^{B'\bar B'} \otimes I^{R\bar R})] \\
&= \int dV^B\ {\rm Tr}[(V^B\sigma^{NB}V^{B\dagger} \otimes V^{\bar B}\sigma^{\bar N\bar B}V^{\bar B\dagger})(F^{N\bar N} \otimes F^{B'\bar B'} \otimes I^{R\bar R})] \\
&= \int dV^B\ {\rm Tr}\Big[(\sigma^{NB} \otimes \sigma^{\bar N\bar B})\Big[(V^{B\dagger} \otimes V^{\bar B\dagger})(I^{R\bar R} \otimes F^{B'\bar B'})(V^B \otimes V^{\bar B})\Big] \otimes F^{N\bar N}\Big] \\
&= {\rm Tr}\Big[(\sigma^{NB} \otimes \sigma^{\bar N\bar B})\Big[\int dV^B\ (V^{B\dagger} \otimes V^{\bar B\dagger})(I^{R\bar R} \otimes F^{B'\bar B'})(V^B \otimes V^{\bar B})\Big] \otimes F^{N\bar N}\Big] \\
&= {\rm Tr}\Big[(\sigma^{NB} \otimes \sigma^{\bar N\bar B})(\alpha I^{B\bar B} + \beta F^{B\bar B}) \otimes F^{N\bar N}\Big] \\
&= \alpha{\rm Tr}[(\sigma^N)^2] + \beta{\rm Tr}[(\rho^{NB})^2]
\end{aligned}$$

where $\bar B$ was an auxiliary copy of the system B. α and β are found using the same method as in previous section

$$\alpha = \frac{|R||B| - |B'|}{|B|^2 - 1} \le \frac{1}{|B'|} \qquad\qquad \beta = \frac{|B'||B| - |R|}{|B|^2 - 1} \le \frac{1}{|R|}$$

Similarly,

$$\begin{aligned}
\int dV^B\ {\rm Tr}[(\sigma^{\rm N}(V^B))^2] &= {\rm Tr}\Big[(\sigma^{NB} \otimes \sigma^{\bar N\bar B})\Big[\int dV^B\ (V^{B\dagger} \otimes V^{\bar B\dagger})(I^{R\bar R} \otimes I^{B'\bar B'})(V^B \otimes V^{\bar B})\Big] \otimes F^{N\bar N}\Big] \\
&= {\rm Tr}\Big[(\sigma^{NB} \otimes \sigma^{\bar N\bar B})(I^{B\bar B} \otimes F^{N\bar N})\Big] \\
&= {\rm Tr}[(\sigma^{\rm N}(V^B))^2]
\end{aligned}$$

Therefore,

$$\begin{aligned}\int dV^B \,\|\sigma^{\text{NB'}}(V^B) - \sigma^N(V^B)\otimes\sigma^{B'}_{\max}\|_2^2 &\leq \int dV^B \left(\text{Tr}[(\sigma^{\text{NB'}}(V^B))^2] - \frac{1}{|B'|}\text{Tr}[(\sigma^N(V^B))^2]\right) \\ &\leq \frac{1}{|B'|}\text{Tr}[(\sigma^N(V^B))^2] + \frac{1}{|R|}\text{Tr}[(\rho^{NB})^2] - \frac{1}{|B'|}\text{Tr}[(\sigma^N(V^B))^2] \\ &\leq \frac{1}{|R|}\text{Tr}[(\rho^{NB})^2]\end{aligned}$$

From Cauchy-Schwarz inequality,

$$\|X\|_1^2 \leq |X|\;\|X\|_2^2$$

$$\begin{aligned}\Rightarrow \int dV^B \,\|\sigma^{\text{NB'}}(V^B) - \sigma^N(V^B)\otimes\sigma^{B'}_{\max}\|_1^2 &\leq \frac{|NB|}{|R|}\int dV^B \,\|\sigma^{\text{NB'}}(V^B) - \sigma^N(V^B)\otimes\sigma^{B'}_{\max}\|_2^2 \\ &\leq \frac{|NB|}{|R|^2}\text{Tr}[(\rho^{NB})^2]\end{aligned}$$

B is maximally entangled with NE, and therefore BN is maximally mixed on a system of dimension $|B|/|N|$, implying $\text{Tr}\big[(\rho^{NB})^2] = |N|/|B|$. Thus,

$$\int dV^B \,\|\sigma^{\text{NB'}}(V^B) - \sigma^N(V^B)\otimes\sigma^{B'}_{\max}\|_1^2 \leq \frac{|N|^2}{|R|^2} = 2^{2(k-s)}$$

Hence, we see that as soon as Bob receives k qubits from the black hole radiation, he gets hold of almost all information that Alice had.

Remark (Trace distance)**.** *The trace distance $\|\rho-\sigma\|_1$ is a bound on how well can two quantum states be distinguished by a generalized measurement (POVM), and hence is a good measure of distinguishability of two states.*

Even though Bob now hold's Alice's information in a subsystem M' of RE, it may be very diffusely distributed within RE. Bob can then perform a computation that maps M' to a compact localized system $\hat{M}$ so that $\rho^{\hat{M}N}$ is a maximally entangled state $|\Phi^{\hat{M}N}\rangle$. The fidelity, which is a measure of correlation of $\rho^{\hat{M}N}$ with $|\Phi^{\hat{M}N}\rangle$, is bounded by

$$F(V^B) \equiv \langle\Phi^{\hat{M}N}|\rho^{\hat{M}N}|\Phi^{\hat{M}N}\rangle \geq 1 - \|\sigma^{\text{NB'}}(V^B) - \sigma^N(V^B)\otimes\sigma^{B'}_{\max}\|_1 \sim 1 - 2^{k-s}$$

Thus, the state obtained after computation is very close to maximally entangled. To prove the above inequality, we start with the definition of fidelity

$$F(\varrho,\varsigma) = \left(\text{Tr}\sqrt{\varrho^{1/2}\varsigma\varrho^{1/2}}\right)^2 = \|\varrho^{1/2}\varsigma^{1/2}\|_1^2$$

$$\text{Tr}(\sqrt{M^\dagger M}) = \sum_i |m_i| \geq \sum_i m_i \geq \text{tr}(M) \Rightarrow \text{Tr}\sqrt{\varrho^{1/2}\varsigma\varrho^{1/2}} \geq \text{Tr}(\sqrt{\varrho}\sqrt{\varsigma}) \Rightarrow \sqrt{F(\varrho,\varsigma)} \geq \text{Tr}(\sqrt{\varrho}\sqrt{\varsigma})$$

$$\Rightarrow \|\sqrt{\varrho}-\sqrt{\varsigma}\|_2^2 = \text{Tr}[(\sqrt{\varrho}-\sqrt{\varsigma})^2] = 2 - 2\text{Tr}(\sqrt{\varrho}\sqrt{\varsigma}) \geq 2 - 2\sqrt{F(\varrho,\varsigma)}$$

$$\varrho - \varsigma = \frac{1}{2}(\sqrt{\varrho}-\sqrt{\varsigma})(\sqrt{\varrho}+\sqrt{\varsigma}) + \frac{1}{2}(\sqrt{\varrho}+\sqrt{\varsigma})(\sqrt{\varrho}-\sqrt{\varsigma})$$

Now consider the basis $|i\rangle$ that diagonalizes $\sqrt{\varrho}-\sqrt{\varsigma}$ with eigenvalues λ_i and U the unitary transformation $U = \sum_i \text{sign}(\lambda_i)|i\rangle\langle i|$.

$$\begin{aligned}\text{Tr}[\,|\varrho-\varsigma|\,] &\geq \text{Tr}[(\varrho-\varsigma)U] \qquad \text{(True for any unitary U)} \\ &= \text{Tr}[\,|\sqrt{\varrho}-\sqrt{\varsigma}|(\sqrt{\varrho}+\sqrt{\varsigma})\,] = \sum_i |\lambda_i|\langle i|\sqrt{\varrho}+\sqrt{\varsigma}|i\rangle \\ &\geq \sum_i |\lambda_i|\,|\langle i|\sqrt{\varrho}-\sqrt{\varsigma}|i\rangle| = \sum_i |\lambda_i|^2 = \|\sqrt{\varrho}-\sqrt{\varsigma}\|_2^2\end{aligned}$$

where in the last line we have used the property that ϱ and ς are positive semi-definite Hermitian operators. Therefore,

$$\mathrm{Tr}[\,|\varrho-\varsigma|\,] \geq 2-2\sqrt{F(\varrho,\varsigma)} \qquad \Rightarrow \sqrt{F(\varrho,\varsigma)} \geq 1-\frac{1}{2}\mathrm{Tr}[\,|\varrho-\varsigma|\,] \qquad \Rightarrow F(\varrho,\varsigma) \geq 1-\mathrm{Tr}[\,|\varrho-\varsigma|\,]$$

$\Psi^{B'RNE}$ is the purification of NB' density operator $\sigma^{NB'}$. If N and B' are decoupled, then ER can be split into two subsystems $E=\hat{M}\check{M}$ such that $\hat{M}$ purifies σ^{N} and $\check{M}$ purifies $\sigma^{B'}$

$$\Psi^{B'RNE} = \Phi^{N\hat{M}} \otimes \Phi^{\check{M}B'}$$

By Uhlmann's Theorem, $F(\varrho,\varsigma)=\max|\langle\psi_{\varrho}|\psi_{\varsigma}\rangle|^2$ where max is over all purifications of ϱ. As $|\Phi^{B'RNE}\rangle = \Phi^{N\hat{M}}\otimes\Phi^{\check{M}B'}$ is the purification of $\sigma^{N}(V^B)\otimes\sigma^{B'}_{\max}$, there exists a purification $|\rho^{B'RNE}\rangle$ of the state $\sigma^{\mathrm{NB'}}(V^B)$, such that

$$|\langle\Phi^{B'RNE}|\rho^{B'RNE}\rangle|^2 = F(\Phi^{B'RNE},\rho^{B'RNE}) \geq 1-\|\sigma^{\mathrm{NB'}}(V^B)-\sigma^{N}(V^B)\otimes\sigma^{B'}_{\max}\|_1$$

As a corollary to Uhlmann's theorem, we have $F(\varrho_{AB},\varsigma_{AB}) \leq F(\varrho_A,\varsigma_A)$ because purifications of AB are also purifications of A. Hence, performing a partial trace on the subsystem $\check{M}B'$ in the above inequality, we obtain

$$\begin{aligned}\langle\Phi^{\hat{M}N}|\rho^{\hat{M}N}|\Phi^{\hat{M}N}\rangle &= F(\mathrm{Tr}_{\check{M}B'}[\Phi^{B'RNE}],\mathrm{Tr}_{\check{M}B'}[\rho^{B'RNE}]) \\ &\geq F(\Phi^{B'RNE},\rho^{B'RNE}) \\ &\geq 1-\|\sigma^{\mathrm{NB'}}(V^B)-\sigma^{N}(V^B)\otimes\sigma^{B'}_{\max}\|_1\end{aligned}$$

Remark (Fidelity)**.** *An equivalent definition of fidelity is* $F(\rho,\sigma)=min_{\{F_i\}}\sum_i\sqrt{Tr[\rho F_i]\,Tr[\sigma F_i]}$*, where the set* $\{F_i\}$ *constitutes a POVM. If the given state is* ρ*, outcome* i *will have probability* $Tr[\rho F_i]$*, and if the given state is* σ*, outcome* i *will have probability* $Tr[\sigma F_i]$*. The fidelity is thus the correlation of the two probability distributions.*

For the case when $|H| >> |E|$, essentially no information is released until the black hole has radiated enough so that $|B'|$ equals $|NRE|$, because till this time the radiation is maximally entangled with the black hole and N may be coupled to B'. But soon after this state is reached, the above analysis becomes applicable and black hole starts radiating the information that Alice had.

4 Conclusion

We observe that if the black hole has radiated more than half of it's initial state, any k qubit information that is thrown into the black hole gets reflected back in the next k qubits the black hole emits, assuming the black hole mixing happens rapidly. In case the black hole is new and hasn't radiated enough, it emits the information in the next k qubits past it's half evaporation.

References

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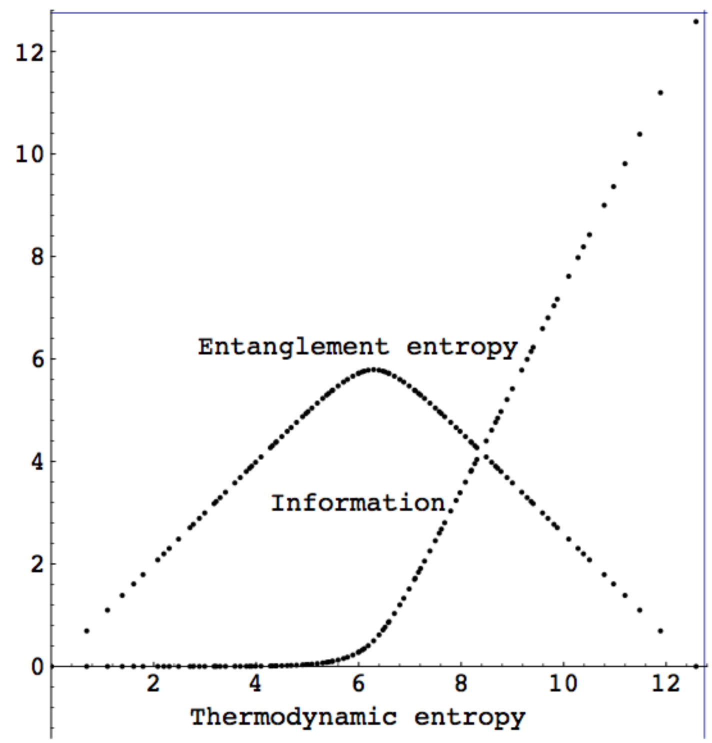

Figure 1: Entanglement entropy and information in a bipartite system. Taken from [2]. Thermodynamic entropy is $\log m$ where m is the size of the system.