Brownian Motion

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Abstract

In these notes we cover the basics of Brownian motion and Ito's Calculus.

1 Stochastic Processes

Definition 1 (Bownian Motion). *A brownian motion is a stochastic process $W_{t\geq 0}$ such that the following properties hold:*

- $W_0 = 0$ *with probability 1.*
- $W_a - W_b \sim N(0, |a-b|)$ *where* $N(m, v)$ *is a Normal distribution with mean* m *and variance* v.
- $W_a - W_b$ *and* $W_c - W_d$ *are independent random variables for* $a > b \geq c > d$.

With parameter t being the time, these properties mean that the process starts with 0 at time 0, and in each time interval the value of the process changes stochastically by a normal distribution centered at 0 and variance equal to the length of the time interval. Increments over two disjoint time intervals are independent.

Definition 2 (σ-Algebra). *A* σ*-algebra on a set* X *is a collection* Σ *of subsets of* X *such that the following properties hold:*

- $X \in \Sigma$.
- $A \in \Sigma \Rightarrow A^c \in \Sigma$ *(closed under complement)*
- $A_i \in \Sigma \Rightarrow \bigcup_i A_i \in \Sigma$ *(closed under countable union)*

In probability theory, probability is a map $\Sigma \to \mathbb{R}$ i.e. it assigns a probability number to each element of the σ-algebra. Each element of the sigma algebra is an event of which we know the probability.

Definition 3 (Filtration). *A filtration is an indexed* σ*-algebra* $\mathcal{F}_t$ *of subsets of the probability sample space such that* $F_a \supseteqq F_b$ *for* $a \geq b$.

Filtration can be understood as information available at time t. The finer the sigma algebra the more events we have there. The condition states that available information grows with time in the filtration.

Definition 4 (Measurable Function). *Given a* σ*-algebra* $\mathcal{F}$ *on the sample space* Ω*, a real-valued random variable* X *is* $\mathcal{F}$*-measurable if the* X*-inverse of any open set in* $\mathbb{R}$ *is contained in* $\mathcal{F}$ *i.e.*

$$\{A \in \Omega | X(A) \in (a, b)\} \in \mathcal{F} \qquad \forall a, b \in \mathbb{R} \quad a > b$$

This means that by observing the value of a measurable function we can point out an event in the sigma algebra which contains all the points in the sample space that would have led to this outcome.

Definition 5 (Adapted Process). *A stochastic process* X_t *is adapted to filtration* $\mathcal{F}_t$ *if for each t,* X_t *is* F_t *measurable.*

Informally this implies that given the information at time t (i.e. for each event E in $\mathcal{F}_\sqcup$ we know whether the outcome is in E or not), X_t is known.

Definition 6 (Generated Filtration)**.** *The filtration generated by a stochastic process is the smallest filtration to which the process is adapted.*

Example 1 Let's take an example of 3 unbiased coin tosses with process X_t counting the number of heads until toss t. The probability sample space Ω is

$$\{\{HHH\},\{HHT\},\{HTH\},\{HTT\},\{THH\},\{THT\},\{TTH\},\{TTT\}\}$$

The filtration generated by the process is

$$\begin{aligned}
\mathcal{F}_0 =\{ \quad & \{\{HHH\},\{HHT\},\{HTH\},\{HTT\},\{THH\},\{THT\},\{TTH\},\{TTT\}\}, && (X_0 = 0)\\
& \emptyset \\
\} \\
\mathcal{F}_1 =\{ \quad & \{\{HHH\},\{HHT\},\{HTH\},\{HTT\}\}, && (X_0 = 0, X_1 = 1)\\
& \{\{THH\},\{THT\},\{TTH\},\{TTT\}\}, && (X_0 = 0, X_1 = 0)\\
& \emptyset \\
& \text{and all unions of the above} \\
\} \\
\mathcal{F}_2 =\{ \quad & \{\{HHH\},\{HHT\}\}, && (X_0 = 0, X_1 = 1, X_2 = 2)\\
& \{\{HTH\},\{HTT\}\}, && (X_0 = 0, X_1 = 1, X_2 = 1)\\
& \{\{THH\},\{THT\}\}, && (X_0 = 0, X_1 = 0, X_2 = 1)\\
& \{\{TTH\},\{TTT\}\}, && (X_0 = 0, X_1 = 0, X_2 = 0)\\
& \emptyset \\
& \text{and all unions of the above} \\
\} \\
\mathcal{F}_3 =\{ \quad & \{\{HHH\}\}, && (X_0 = 0, X_1 = 1, X_2 = 2, X_3 = 3)\\
& \{\{HHT\}\}, && (X_0 = 0, X_1 = 1, X_2 = 2, X_3 = 2)\\
& \{\{HTH\}\}, && (X_0 = 0, X_1 = 1, X_2 = 1, X_3 = 2)\\
& \{\{HTT\}\}, && (X_0 = 0, X_1 = 1, X_2 = 1, X_3 = 1)\\
& \{\{THH\}\}, && (X_0 = 0, X_1 = 0, X_2 = 1, X_3 = 2)\\
& \{\{THT\}\}, && (X_0 = 0, X_1 = 0, X_2 = 1, X_3 = 1)\\
& \{\{TTH\}\}. && (X_0 = 0, X_1 = 0, X_2 = 0, X_3 = 1)\\
& \{\{TTT\}\}, && (X_0 = 0, X_1 = 0, X_2 = 0, X_3 = 0)\\
& \emptyset \\
& \text{and all unions of the above} \\
\}
\end{aligned}$$

Definition 7 (Expectation conditional on filtration)**.** $\mathbb{E}^{\mathbb{P}}[X|\mathcal{F}]$ *is a* $\mathcal{F}$ *measurable random variable such that*

$$\mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}[X|\mathcal{F}]|A] = \mathbb{E}^{\mathbb{P}}[X|A] \quad \forall A \in \mathcal{F} \tag{1}$$

In the coin toss example above,

$$\mathbb{E}[X_3|\mathcal{F}_1](s) = \begin{cases} 2 & \text{if} \quad s \in \{\{HHH\},\{HHT\},\{HTH\},\{HTT\}\} \\ 1 & \text{if} \quad s \in \{\{THH\},\{THT\},\{TTH\},\{TTT\}\} \end{cases}$$

Definition 8 (Martingale)**.** *A stochastic process X_t is a martingale with respect to a filtration $\mathcal{F}_t$ of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ if*

$$\mathbb{E}^{\mathbb{P}}[X_a|\mathcal{F}_b] = X_b \quad for \ a \geq b$$

Martingale property implies that expected value of the process at any future time a is equal to the value at present time b.

Exercise. *Show that in the coin toss example, if we had instead counted number of heads minus number of tails, X_t would have been a martingale. For example, show that $\mathbb{E}[X_3|\mathcal{F}_1]$ would have been equal to X_1.*

Definition 9 ($\mathcal{F}_t$ Brownian motion)**.** *A Brownian motion W_t is a $\mathcal{F}_t$ Brownian motion if W_t is adapted to $\mathcal{F}_t$ and $W_{t+s} - W_t$ is independent of $\mathcal{F}_t$ for $s > 0$.*

Remark (An $\mathcal{F}_t$ Brownian motion is a martingale with respect to filtration $\mathcal{F}_t$)**.**

$$\begin{aligned}
\mathbb{E}[W_a|\mathcal{F}_b] &= \mathbb{E}[W_a - W_b|\mathcal{F}_b] + \mathbb{E}[W_b|\mathcal{F}_b] \\
&= 0 + \mathbb{E}[W_b|\mathcal{F}_b] \\
&= W_b
\end{aligned}$$

2 Ito Calculus

Definition 10 (Ito Integral)**.** *Let H_t be a $\mathcal{F}_t$ adapted process and W_t be a $\mathcal{F}_t$ Brownian motion. The Ito integral is defined as*

$$\int_0^t H_s dW_s = \lim_{n\to\infty} \sum_{i=0}^{2^n-1} H_{it/2^n}(W_{(i+1)t/2^n} - W_{it/2^n})$$

This is similar to the Riemann integral. One thing to note here is that the integrand always takes the value at the start of the interval. If we make the integrand take the the average of the values at interval endpoints, we get

Definition 11 (Stratonovich Integral)**.**

$$\int_0^t H_s \circ dW_s = \lim_{n\to\infty} \sum_{i=0}^{2^n-1} (H_{it/2^n} + H_{it/2^n})(W_{(i+1)t/2^n} - W_{it/2^n})/2 \tag{2}$$

The two integrals usually do not give the same resutls, as can be checked by repeating the calculations we do in these notes for Stratonovich integral as well.

In finance the integrand is usually the quantities of assets in the portfolio and the stochastic process the price of these assets. When re-adjusting the portfolio for a specific time interval we can only use the information available at the beginning of the time interval and therefore using Ito integral would be more appropriate.

Let's compute expectations and variances of some of the basic Ito Integrals. We assume that the integrand is square-intregrable i.e. $\int_0^t \mathbb{E}[H_s^2]ds < \infty$. In this case we have

Calculation (Expectation of Ito Integral)**.**

$$
\begin{aligned}
\mathbb{E}[\int_0^t H_s dW_s|\mathcal{F}_0] &= \mathbb{E}[\lim_{n\to\infty}\sum_{i=0}^{2^n-1} H_{it/2^n}(W_{(i+1)t/2^n} - W_{it/2^n})|\mathcal{F}_{\prime}] && \textit{(by definition)}\\
&= \lim_{n\to\infty}\sum_{i=0}^{2^n-1} \mathbb{E}[H_{it/2^n}(W_{(i+1)t/2^n} - W_{it/2^n})|\mathcal{F}_0] && \textit{(by square integrability of } H_s\textit{)}\\
&= \lim_{n\to\infty}\sum_{i=0}^{2^n-1} \mathbb{E}[\mathbb{E}[H_{it/2^n}(W_{(i+1)t/2^n} - W_{it/2^n})|\mathcal{F}_{it/2^n}]|\mathcal{F}_0] && \textit{(iterated conditioning)}\\
&= \lim_{n\to\infty}\sum_{i=0}^{2^n-1} \mathbb{E}[H_{it/2^n}\mathbb{E}[(W_{(i+1)t/2^n} - W_{it/2^n})|\mathcal{F}_{it/2^n}]|\mathcal{F}_0] && \textit{(} H_s \textit{ is adapted to } \mathcal{F}_s\textit{)}\\
&= \lim_{n\to\infty}\sum_{i=0}^{2^n-1} \mathbb{E}[H_{it/2^n}0|\mathcal{F}_0] && \textit{(} W_s \textit{ is } \mathcal{F}_s \textit{ Brownian)}\\
&= 0
\end{aligned}
$$

By iterated conditioning we also have $\mathbb{E}[\int_0^t H_s dW_s] = \mathbb{E}[\mathbb{E}[\int_0^t H_s dW_s|\mathcal{F}_0]] = \mathbb{E}[0] = 0$.

Calculation (Covariance of Ito Integral)**.**

$$\mathbb{E}[\int_0^t H_s dW_s \int_0^t K_s dW_s|\mathcal{F}_0]$$

$$= \mathbb{E}[\lim_{n,m\to\infty} \sum_{i,j=0}^{2^n-1,2^m-1} H_{it/2^n}(W_{(i+1)t/2^n} - W_{it/2^n})K_{jt/2^m}(W_{(j+1)t/2^m} - W_{jt/2^m})|\mathcal{F}_0]$$

(by definition)

$$= \lim_{n,m\to\infty} \sum_{i,j=0}^{2^n-1,2^m-1} \mathbb{E}[H_{it/2^n}(W_{(i+1)t/2^n} - W_{it/2^n})K_{jt/2^m}(W_{(j+1)t/2^m} - W_{jt/2^m})|\mathcal{F}_0]$$

(by square integrability of H_s)

$$= \lim_{n,m\to\infty} \sum_{i,j=0}^{2^n-1,2^m-1} \mathbb{E}[\mathbb{E}[H_{it/2^n}(W_{(i+1)t/2^n} - W_{it/2^n})K_{jt/2^m}(W_{(j+1)t/2^m} - W_{jt/2^m})|\mathcal{F}_{\max(i,j)t/2^n}]|\mathcal{F}_0]$$

(iterated conditioning)

$$= \lim_{n\to\infty} \sum_{i,j=0}^{2^n-1} \mathbb{E}[\mathbb{E}[H_{it/2^n}(W_{(i+1)t/2^n} - W_{it/2^n})K_{jt/2^n}(W_{(j+1)t/2^n} - W_{jt/2^n})|\mathcal{F}_{\max(i,j)t/2^n}]|\mathcal{F}_0]$$

(assuming same partition to simplify algebra)

$$= \lim_{n\to\infty} \sum_{i,j=0}^{2^n-1} \begin{cases} \mathbb{E}[H_{it/2^n}K_{jt/2^n}(W_{(j+1)t/2^n} - W_{jt/2^n})\mathbb{E}[(W_{(i+1)t/2^n} - W_{it/2^n})|\mathcal{F}_{it/2^n}]|\mathcal{F}_0] & \text{if} \quad i > j \\ \mathbb{E}[H_{it/2^n}K_{it/2^n}\mathbb{E}[(W_{(i+1)t/2^n} - W_{it/2^n})(W_{(i+1)t/2^n} - W_{it/2^n})|\mathcal{F}_{it/2^n}]|\mathcal{F}_0] & \text{if} \quad i = j \\ \mathbb{E}[H_{it/2^n}(W_{(i+1)t/2^n} - W_{it/2^n})K_{jt/2^n}\mathbb{E}[(W_{(j+1)t/2^n} - W_{jt/2^n})|\mathcal{F}_{jt/2^n}]|\mathcal{F}_0] & \text{if} \quad i < j \end{cases}$$

$$= \lim_{n\to\infty} \sum_{i,j=0}^{2^n-1} \begin{cases} \mathbb{E}[H_{it/2^n}K_{jt/2^n}(W_{(j+1)t/2^n} - W_{jt/2^n})0|\mathcal{F}_0] & \text{if} \quad i > j \\ \mathbb{E}[H_{it/2^n}K_{it/2^n}\text{Var}[W_{(i+1)t/2^n} - W_{it/2^n}|\mathcal{F}_{it/2^n}]|\mathcal{F}_0] & \text{if} \quad i = j \\ \mathbb{E}[H_{it/2^n}(W_{(i+1)t/2^n} - W_{it/2^n})K_{jt/2^n}0|\mathcal{F}_0] & \text{if} \quad i < j \end{cases}$$

$$= \lim_{n\to\infty} \sum_{i,j=0}^{2^n-1} \begin{cases} 0 & \text{if} \quad i \neq j \\ \mathbb{E}[H_{it/2^n}K_{it/2^n}t/2^n|\mathcal{F}_0] & \text{if} \quad i = j \end{cases}$$

$$= \lim_{n\to\infty} \sum_{i=0}^{2^n-1} \mathbb{E}[H_{it/2^n}K_{it/2^n}|\mathcal{F}_0]$$

$$= \mathbb{E}[\lim_{n\to\infty} \sum_{i=0}^{2^n-1} H_{it/2^n}K_{it/2^n}|\mathcal{F}_0]$$

$$= \mathbb{E}[\int_0^t H_s K_s ds|\mathcal{F}_0]$$

Remark (Ito Isometry)**.** *The above result is known as Ito isometry. Let* $\mathcal{I}(X) = \int_0^t X_s dW_s$ *i.e. the Ito intregration. Then Ito Integration can be seen as a map from* $L^2_{ad}([0,t]\times\Omega)$ *to* $L^2(\Omega)$ *such that the inner product is preserved i.e.* $(\mathcal{I}(H), \mathcal{I}(K))_{L^2(\Omega)} = (H,K)_{L^2_{ad}([0,t]\times\Omega)}$ *where* $(X,Y)_{L^2(\Omega)} \equiv \mathbb{E}[XY]$ *and* $(X,Y)_{L^2_{ad}([0,t]\times\omega)} \equiv \mathbb{E}[\int_0^t X_t Y_t dt]$.

With $H = K$, the above two results imply

$$\begin{aligned} \mathbb{E}[\int_0^t H_s dW_s] &= 0 \\ \text{Var}[\int_0^t H_s dW_s] &= \mathbb{E}[\int_0^t H_s^2 ds]. \end{aligned}$$

One key feature of Ito calulus is that the second order differentials are not equivalent to 0. In ordinary calculus we have

$$\begin{aligned}
\int_0^t X(t)(dt)^2 &\equiv \lim_{n\to\infty}\sum_{i=0}^{2^n-1} X(it/2^n)((i+1)t/2^n - it/2^n)^2 \\
&= \lim_{n\to\infty}\sum_{i=0}^{2^n-1} X(it/2^n)(t/2^n)^2 \\
&\leq \max_{0<s<t}(X(s)) \lim_{n\to\infty}\sum_{i=0}^{2^n-1}(t/2^n)^2 \\
&\leq \max_{0<s<t}(X(s))t \lim_{n\to\infty}\sum_{i=0}^{2^n-1}(t/2^n) \\
&= 0
\end{aligned}$$

In stochastic calculus we can check the expectation and variance of the distribution of the integral:

Calculation (Integral of second order time differential)**.**

$$\begin{aligned}
\mathbb{E}[\int_0^t X_t(dt)^2] &\equiv \mathbb{E}[\lim_{n\to\infty}\sum_{i=0}^{2^n-1} X_{it/2^n}((i+1)t/2^n - it/2^n)^2] \\
&= \lim_{n\to\infty}\sum_{i=0}^{2^n-1}\mathbb{E}[X_{it/2^n}](t/2^n)^2 \\
&\leq \max_{0<s<t}(|\mathbb{E}[X_s]|) \lim_{n\to\infty}\sum_{i=0}^{2^n-1}(t/2^n)^2 \\
&\leq \max_{0<s<t}(|\mathbb{E}[X_s]|)t \lim_{n\to\infty}\sum_{i=0}^{2^n-1}(t/2^n) \\
&= 0
\end{aligned}$$

Variance can be similarly computed to be 0 if the stochastic process has finite mean and variance at all times. Therefore the distrubution is 0 with probability 1.

However the integral does not necessarily vanish when taken with respect to second order differential of Brownian motion. We have

Calculation (Expectation of integral of second order Brownian motion differential).

$$
\begin{aligned}
\mathbb{E}[\int_0^t H_s(dW_s)^2|\mathcal{F}_0] &= \mathbb{E}[\lim_{n\to\infty}\sum_{i=0}^{2^n-1} H_{it/2^n}(W_{(i+1)t/2^n} - W_{it/2^n})^2|\mathcal{F}_/] \\
&= \lim_{n\to\infty}\sum_{i=0}^{2^n-1} \mathbb{E}[H_{it/2^n}(W_{(i+1)t/2^n} - W_{it/2^n})^2|\mathcal{F}_0] \\
&= \lim_{n\to\infty}\sum_{i=0}^{2^n-1} \mathbb{E}[\mathbb{E}[H_{it/2^n}(W_{(i+1)t/2^n} - W_{it/2^n})^2|\mathcal{F}_{it/2^n}]|\mathcal{F}_0] \\
&= \lim_{n\to\infty}\sum_{i=0}^{2^n-1} \mathbb{E}[H_{it/2^n}\mathbb{E}[(W_{(i+1)t/2^n} - W_{it/2^n})^2|\mathcal{F}_{it/2^n}]|\mathcal{F}_0] \\
&= \lim_{n\to\infty}\sum_{i=0}^{2^n-1} \mathbb{E}[H_{it/2^n}t/2^n|\mathcal{F}_0] \\
&= \mathbb{E}[\int_0^t H_s dt|\mathcal{F}_0]
\end{aligned}
$$

We therefore have $\mathbb{E}[\int_0^t H_s(dW_s)^2 - \int_0^t H_s dt] = 0$. So the two integrals have same expectation. Let's see if their difference has any variance.

Calculation (Variance of integral of second order Brownian motion differential)**.**

$$\mathbb{E}[\big(\int_0^t H_s(dW_s)^2 - \int_0^t H_s dt\big)^2|\mathcal{F}_0]$$
$$= \mathbb{E}[\lim_{n\to\infty}\big(\sum_{i=0}^{2^n-1} H_{it/2^n}(W_{(i+1)t/2^n} - W_{it/2^n})^2 - ((i+1)t/2^n - it/2^n)\big)^2|\mathcal{F}_0]$$
$$= \mathbb{E}[\lim_{n\to\infty}\big(\sum_{i=0}^{2^n-1} H_{it/2^n}(W_{(i+1)t/2^n} - W_{it/2^n})^2 - t/2^n\big)^2|\mathcal{F}_0]$$
$$= \mathbb{E}[\lim_{n\to\infty}\sum_{i,j=0}^{2^n-1} H_{it/2^n}H_{jt/2^n}((W_{(i+1)t/2^n} - W_{it/2^n})^2 - t/2^n)((W_{(j+1)t/2^n} - W_{jt/2^n})^2 - t/2^n)|\mathcal{F}_0]$$
$$= \lim_{n\to\infty}\sum_{i,j=0}^{2^n-1} \mathbb{E}[H_{it/2^n}H_{jt/2^n}((W_{(i+1)t/2^n} - W_{it/2^n})^2 - t/2^n)((W_{(j+1)t/2^n} - W_{jt/2^n})^2 - t/2^n)|\mathcal{F}_0]$$
$$= \lim_{n\to\infty}\sum_{i,j=0}^{2^n-1} \mathbb{E}[\mathbb{E}[H_{it/2^n}H_{jt/2^n}((W_{(i+1)t/2^n} - W_{it/2^n})^2 - t/2^n)((W_{(j+1)t/2^n} - W_{jt/2^n})^2 - t/2^n)|\mathcal{F}_{\max(i,j)t/2^n}]|\mathcal{F}_0]$$
$$= \lim_{n\to\infty}\sum_{i,j=0}^{2^n-1}\begin{cases}\mathbb{E}[H_{it/2^n}H_{jt/2^n}(W_{(j+1)t/2^n} - W_{jt/2^n})^2 - t/2^n)\mathbb{E}[(W_{(i+1)t/2^n} - W_{it/2^n})^2 - t/2^n|\mathcal{F}_{it/2^n}]|\mathcal{F}_0] & \textit{if} \quad i > j \\ \mathbb{E}[(H_{it/2^n})^2\mathbb{E}[((W_{(i+1)t/2^n} - W_{it/2^n})^2 - t/2^n)^2|\mathcal{F}_{it/2^n}]|\mathcal{F}_0] & \textit{if} \quad i = j \\ \mathbb{E}[H_{it/2^n}H_{jt/2^n}(W_{(i+1)t/2^n} - W_{it/2^n})^2 - t/2^n)\mathbb{E}[(W_{(j+1)t/2^n} - W_{jt/2^n})^2 - t/2^n|\mathcal{F}_{jt/2^n}]|\mathcal{F}_0] & \textit{if} \quad i < j\end{cases}$$
$$= \lim_{n\to\infty}\sum_{i,j=0}^{2^n-1}\begin{cases}\mathbb{E}[H_{it/2^n}H_{jt/2^n}(W_{(j+1)t/2^n} - W_{jt/2^n})^2 - t/2^n)0|\mathcal{F}_0] & \textit{if} \quad i > j \\ \mathbb{E}[(H_{it/2^n})^2\mathbb{E}[(W_{(i+1)t/2^n} - W_{it/2^n})^4 + (t/2^n)^2 - 2(W_{(i+1)t/2^n} - W_{it/2^n})^2 t/2^n|\mathcal{F}_{it/2^n}]|\mathcal{F}_0] & \textit{if} \quad i = j \\ \mathbb{E}[H_{it/2^n}H_{jt/2^n}(W_{(i+1)t/2^n} - W_{it/2^n})^2 - t/2^n)0|\mathcal{F}_0] & \textit{if} \quad i < j\end{cases}$$
$$= \lim_{n\to\infty}\sum_{i,j=0}^{2^n-1}\begin{cases}0 & \textit{if} \quad i \neq j \\ \mathbb{E}[(H_{it/2^n})^2(3(t/2^n)^2 + (t/2^n)^2 - 2(t/2^n)^2)|\mathcal{F}_0] & \textit{if} \quad i = j\end{cases}$$
$$= \lim_{n\to\infty}\sum_{i=0}^{2^n-1}\mathbb{E}[(H_{it/2^n})^2(2(t/2^n)^2)|\mathcal{F}_0]$$
$$= \int_0^t 2\mathbb{E}[(H_s)^2|\mathcal{F}_0](dt)^2$$
$$= 0$$

We therefore have $\mathbb{E}[\int_0^t H_s(dW_s)^2 - \int_0^t H_s dt] = 0$ and $\mathrm{Var}[\int_0^t H_s(dW_s)^2 - \int_0^t H_s dt] = 0$ and therefore $\mathbb{P}(\int_0^t H_s(dW_s)^2 = \int_0^t H_s dt) = 1$ i.e.

$$\int_0^t H_s(dW_s)^2 = \int_0^t H_s dt \quad \text{almost surely}$$

Remark (Ito Differentials)**.** *The above result is also written in differential notation as* $(dW)^2 = dt$*. We can similarly check that* $dtdW = 0$ *and* $(dW)^3 = 0$*. Therefore we have the algebra on differentials:*

$$(dt)^2 = 0$$
$$(dW)^2 = dt$$

From the ordinary calculus we have the Taylor expansion of a function as

$$f((x) + \boldsymbol{\delta x}) = f(\boldsymbol{x}) + f'(\boldsymbol{x}) \cdot \boldsymbol{\delta x} + \frac{1}{2}(\boldsymbol{\delta x})^T \cdot f'' \cdot \boldsymbol{\delta x} + \dots$$

where f' is the gradient and f'' is the Hessian. When taking the limit $\delta x \to dx$ we often ignore the second and higher order differentials $(dx)^2, (dx)^3, \ldots$ giving the usual chain rule. In stochastic calculus, however, as we have seen second order differenials need not be 0. And we have:

Remark (Ito's Lemma)**.**

$$df(\boldsymbol{x}) = f'(\boldsymbol{x}) \cdot \boldsymbol{dx} + (\boldsymbol{dx})^T \cdot f'' \cdot \boldsymbol{dx} \tag{3}$$

and in the special case when f is a function of t and a stochastic process X_t.

$$df(t, X_t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X_t}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}(dX_t)^2$$

and when X_t is a brownian motion W_t we have

$$df(t, X_t) = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial W_t^2}\right)dt + \frac{\partial f}{\partial W_t}dW_t$$

Definition 12 (Ito Diffusion)**.** *An Ito diffusion is a stochastic process satisying a stochastic differential equation of the form*

$$d\boldsymbol{X}_t = \underbrace{\boldsymbol{\mu}(t, \boldsymbol{X}_t)dt}_{drift} + \underbrace{\boldsymbol{\sigma}(t, \boldsymbol{X}_t) \cdot d\boldsymbol{W}_t}_{diffusion} \tag{4}$$

The diffusion coefficient in the equation above is also known as volatility.

Definition 13 (Local Martingale)**.** *An Ito diffusion with zero drift is a local martingale*

$$d\boldsymbol{X}_t = \boldsymbol{\sigma}(t, \boldsymbol{X}_t) \cdot d\boldsymbol{W}_t$$

Example 2 Geometric Brownian Motion

$$\frac{dX_t}{X_t} = \mu dt + \sigma dW_t \tag{5}$$

$$\begin{aligned}
d(\log(X_t) &= \frac{dX_t}{X_t} - \frac{1}{2}\frac{1}{X_t^2}(dX_t)^2 && \text{(by Ito's Lemma)} \\
&= \mu dt + \sigma dW_t - \frac{1}{2}\sigma^2 dt \\
&= \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t \\
\log(X_t) - \log(X_0) &= \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right)ds + \sigma dW_s \\
&= \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t \\
X_t &= X_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}
\end{aligned}$$

Because of its simplicity Geometric Brownian Motion has been widely used in finance as a model for evolution of stock prices.