No-Arbitrage Pricing

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Abstract

In these notes we try to explain what arbitrage is and how to price financial derivates in order to avoid arbitrage.

1 Binomial Model

To get an intuition for the concept of arbitrage and risk neutral pricing, we digress a bit from the continuous time stochastic process model and work for a while on the simpler descreet time binomial model, where in each time step, given the value at initial time, the process can take one of two possible values with certain probabilities at the final time.

Definition 1 (Forward Contract)**.** *A forward contract is an agreement between two parties to buy or sell a financial tradable at a decided date in the future at a price determined at the inception of the contract. The set price is known as the forward price, and the decided date in future is known as the expiration date. Usually there is no initial exchange of cashflows before the expiration.*

Let's start with a simple example. Suppose we have a stock trading in the market for \$100 and from the historical probability we know that in a year the stock may go to \$120 with probability 0.5 and may go to \$90 with probability 0.5. Assume that the bank borrows and lends money at an interest of 2%. Suppose that there is a forward contract trading in the market for the stock which lets you buy or sell the stock a year later. With the given historic probabilities, one may be inclined to price the forward contract at

$$\mathbb{E}^{\mathbb{P}^{\text{historic}}}[S_{\text{1year}}] = 0.5 \cdot \$120 + 0.5 \cdot \$90 = \$105$$

Suppose that a bank is actually trading the forward for \$105. Can you figure out a way to make profit with probability 1?

We can sell the forward derivative for \$105, take a loan from the bank for \$100, use this money to buy the stock right now, give this stock to the buyer of the forward for \$105 a year from now, and repay the \$102 to the bank for the loan we took. We start with no money and at the end of the year we have a profit of \$3 with probability 1. This is arbitrage.

Definition 2 (Arbitrage)**.** *A portfolio is an arbitrage if we have*

$$\pi_0 = 0 \qquad \mathbb{P}^{historic}(\pi_T \geq 0) = 1 \qquad \mathbb{P}^{historic}(\pi_T > 0) > 0 \tag{1}$$

where π_t is the value of the portfolio at time t. The portfolio must be self-financing meaning that we do not put more cash into the portfolio during the execution of the strategy.

The two conditions on probability mean that almost surely we incur no loss from executing our strategy, and that there is a non-zero chance of gaining profit.

Definition 3 (Option contract)**.** *An option contract is an agreement between two parties where the buyer of the contract has the option, but not the obligation, to buy or sell a financial tradable at a decided date in the future at a price determined at the inception of the contract. An option contract with the option to buy (sell) is called a call (put) option, the set price is known as the strike, and the decided date in future is known as the option expiration date.*

Definition 4 (Bond)**.** *A bond is a contract between two parties that pays* \$1 *to the buyer of the bond at the expiration date.*

The riskless bond price is governed primarily by the interest rates.

Example 1 Binomial Option Pricing Let's see how we can price an option in a binomial model so as to avoid arbitrage. Take the same stock in the previous example. Suppose a client wants to buy a call option on the stock with strike as \$105 and expiration in a year. In a year if the stock price goes up to \$120, the client will want to excercise the option and get the stock for \$105, and if the stock price goes down to \$90 the client is better off not exercising the option and simply buying the stock from the market. Note that if we simply buy the stock now to give it to the client in future we are protected againt the risk of stock prices rising if the client chooses to exercise, however if the stock prices go down and the client does not exercies we have the risk of losing money on the stock. So we try to think of a replication portfolio made up of stocks and bonds such that the final payoff of the portfolio is always equal to the payoff of the option at expiration. And if we can buy and buy this replcation portfolio now then no matter what turn the stock price takes we have zero risk. The final payoff

$$
\begin{aligned}
&= \begin{cases} S_{\text{1year}} - \$105 & \text{if} \quad S_{\text{1year}} = \$120 \\ 0 & \text{if} \quad S_{\text{1year}} = \$90 \end{cases} \\
&= \begin{cases} \frac{1}{2} S_{\text{1year}} - \$45 & \text{if} \quad S_{\text{1year}} = \$120 \\ \frac{1}{2} S_{\text{1year}} - \$45 & \text{if} \quad S_{\text{1year}} = \$90 \end{cases} \\
&= \frac{1}{2} S_{\text{1year}} - \$45 \\
&= \frac{1}{2} S_{\text{1year}} - 45 B_{\text{1year}}
\end{aligned}
$$

So the option payoff can indeed be replicated by a portfolio of stocks and bonds. If we buy 1/2 stock and sell 45 bonds then we have completely hedged our risk. The price of this replication portfolio, and hence the call option, is

$$\frac{1}{2} \cdot S_{\text{now}} - 45 \cdot B_{\text{now}} = \frac{1}{2} \cdot \$100 - 45 \cdot \$\frac{1}{1.02} \sim \$6$$

If the price of the call option were any different from the price of the replication portfolio, one could trade in option and replication portflio and make unlimited profits.

Exercise (Binomial Forward Price). *Show that the forward price of the forward contract in the first example should be equal to S_{now}/B_{now} to not have arbitrage.*

Because of the discrete binary branching in the binomial model it is always possible to form a replicating portfolio with just stocks and bond. We simply have to solve a system of two linear equations

$$
\begin{aligned}
a \cdot S_u + b \cdot B &= D_u \\
a \cdot S_d + b \cdot B &= D_d
\end{aligned}
$$

where S_u and S_d are the two possible stock price a year from now, and D_u and D_d are the corresponding payoff of the stock derivative. B is a bond with 1 payoff a year from now. The current price of the derivative is then

$$D_{\text{now}} = a \cdot S_{\text{now}} + b \cdot B_{\text{now}} \tag{2}$$

Sovling for a and b we get

$$
\begin{aligned}
a &= \frac{D_u - D_d}{S_u - S_d} \\
b &= \frac{D_d \cdot S_u - D_u \cdot S_d}{S_u - S_d} \\
D_{\text{now}} &= \frac{D_u - D_d}{S_u - S_d} \cdot S_{\text{now}} + \frac{D_d \cdot S_u - D_u \cdot S_d}{S_u - S_d} \cdot B_{\text{now}} \\
&= \frac{S_{\text{now}} - S_d B_{\text{now}}}{S_u - S_d} \cdot D_u + \frac{-S_{\text{now}} + S_u B_{\text{now}}}{S_u - S_d} \cdot D_u \\
&\equiv q \cdot B_{\text{now}} D_u + (1-q) \cdot B_{\text{now}} D_d
\end{aligned}
$$

Definition 5 (Risk Neutral Probability)**.** *In the final representation of the no-arbitrage derivative pricing formula above, q is sometimes interpreted as a probability measure and is known as the risk neutral probability. The formula can then be interpreted as an expectation under this probability.*

Note that the real world historic probabilities don't show up in the no-arbitrage pricing formula at all. Before getting back into continuous time stochastic model, it is worth noting an identity which has a very meaninful translation in the continuous time version as well.

Remark (Return on Stock in Risk Neutral Measure)**.**

$$\mathbb{E}^{\mathbb{P}-risk\text{-}neutral}[S_{1year}] = q \cdot S_u + (1-q) \cdot S_d = S_{now}/B_{now} \tag{3}$$

i.e. expected return on a stock in the risk neutral probability measure is the same as the return on a riskless bond.

2 Continous Time Ito Diffusion Model

Let's get back to continuous time processes now.

Consider a porfolio of stocks and money market. The value of portfolio at any time is

$$
\begin{aligned}
\pi_t &= \Delta_t S_t + M_t \\
\pi_0 &= 0
\end{aligned}
$$

where Δ_t is the amount of stocks of price S_t held at time t and M_t is the amount invested in money market which gives an instantaneous return of rdt where r is the riskfree interest rate.

$$d_{\text{market}}\pi_t = \Delta_t dS_t + rM_t dt$$

i.e. the change in portfolio value due to market movement. After any time step we neither inject nor withdraw cash from the portfolio, we just rebalance the positions giving us

$$
\begin{aligned}
d\pi_t &= d_{\text{market}}\pi_t \\
&= \Delta_t dS_t + rM_t dt \\
(d\Delta_t)S_t + \Delta_t dS_t + (d\Delta_t)(dS_t) + dM_t &= \Delta_t dS_t + rM_t dt && \text{(using Ito's Lemma)} \\
dM_t &= rM_t dt - (d\Delta_t)(S_t + dS_t)
\end{aligned}
$$

Take the ansatz $M_t = e^{\int_0^t rdt}\tilde{M}_t$ and $S_t = e^{\int_0^t rdt}\tilde{S}_t$. We then have

$$
\begin{aligned}
e^{\int_0^t rdt} d\tilde{M}_t + re^{\int_0^t rdt}\tilde{M}_t dt &= re^{\int_0^t rdt}\tilde{M}_t dt - (d\Delta_t)(e^{\int_0^t rdt}\tilde{S}_t + e^{\int_0^t rdt} d\tilde{S}_t + re^{\int_0^t rdt}\tilde{S}_t dt) \\
d\tilde{M}_t &= -(d\Delta_t)((1 + rdt)\tilde{S}_t + d\tilde{S}_t) \\
&= -(d\Delta_t)(\tilde{S}_t + d\tilde{S}_t) && (\because (d\Delta_t)(dt) = 0)
\end{aligned}
$$

Therefore, with the ansatz $\pi_t = e^{\int_0^t r dt}\tilde{\pi}_t$, we have

$$
\begin{aligned}
\tilde{\pi}_t &= \Delta_t \tilde{S}_t + \tilde{M}_t \\
d\tilde{\pi}_t &= d(\Delta_t \tilde{S}_t) + d\tilde{M}_t \\
&= d\Delta_t d\tilde{S}_t + (d\Delta_t)\tilde{S}_t + \Delta_t d\tilde{S}_t + d\tilde{M}_t \\
&= \Delta_t d\tilde{S}_t
\end{aligned}
$$

Denoting $e^{\int_0^t r dt}$ by A_t, also known as the accumulation factor, we have

$$\frac{\pi_T}{A_T} = \int_0^T \Delta_t d\frac{S_t}{A_t}$$

We can take any price of any tradable asset, including derivatives of stocks, as S_t and the analysis still holds. $D_t = 1/A_t$ is also known as discount factor and $D_t S_t = S_t/A_t$ is known as the discounted asset price.

Definition 6 (Equivalent Measures)**.** *Two probability measures $\mathbb{P}^a$ and $\mathbb{P}^b$ are equivalent if for any event A of the σ-algebra $\mathbb{P}^a(A) = 0$ if and only if $\mathbb{P}^b(A) = 0$.*

Lemma 7 (Sufficient condition for no-arbitrage)**.** *If there is a measure $\mathbb{P}$ equivalent to the real world historic measure $\mathbb{P}^{historic}$ such that the discounted tradable asset process S_t/A_t is a local martingale under $\mathbb{P}$, then the market $\{S_t, M_t\}$ has no arbitrage.*

We show that the two conditions of arbitrage

$$\mathbb{P}^{\text{historic}}(\pi_T \geq 0) = 1 \qquad \mathbb{P}^{\text{historic}}(\pi_T > 0) > 0$$

cannot be simultaneously satisfied. Since $\mathbb{P}$ and $\mathbb{P}^{\text{historic}}$ are equivalent measures we get the equivalent conditions

$$\mathbb{P}(\pi_T \geq 0) = 1 \qquad \mathbb{P}(\pi_T > 0) > 0$$

As $\pi_T/A_T = \int_0^T \Delta_t d(S_t/A_t)$ is an Ito Integral when (S_t/A_t) is a local martingale, and we can assume the bounded regularity conditions on Δ_t, we have

$$\mathbb{E}^{\mathbb{P}}[\pi_T/A_T] = \pi_0 = 0$$

Combining with the first condition, $\mathbb{P}(\pi_T \geq 0) = 1$ we have $\mathbb{P}(\pi_T = 0) = 1$ as $A_T > 0$, contradicting the second condition $\mathbb{P}(\pi_T > 0) > 0$.

The converse of this lemma is quite hard to prove. We will get to back to it later. Together they give the Fundamental Theorem of Asset Pricing.

Theorem (First Fundamental Theorem of Asset Pricing)**.** *The market $\{S_t, M_t\}$ has no arbitrage if and only if there is a measure $\mathbb{P}$ equivalent to the real world historic measure $\mathbb{P}^{historic}$ such that the discounted tradable asset process S_t/A_t is a local martingale under $\mathbb{P}$.*