Black Scholes

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Abstract

In these notes we price an option contract using the no-arbitrage pricing theory.

1 Black Scholes Option Pricing

Definition 1 (Black Scholes Model)**.** *Black Scholes model is model for the market where the risk-free interest rate r is constant and tradable asset price process follows Geometric Brownian motion i,e.*

$$D_t \sim e^{-rt}$$
$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

We assume that the no-arbitrage condition holds in the market and that there is a risk neutral probability measure $\mathbb{P}$ such that discounted asset process is a local martingale under this measure.

$$\begin{aligned} d(D_tS_t)_{\text{drift}} &= 0 \\ \big(-rD_tS_tdt + D_tS_t(\mu dt + \sigma dW_t^{\mathbb{P}}) + 0\big)_{\text{drift}} &= 0 \\ (\mu - r)D_tS_t &= 0 \end{aligned}$$

Therefore we have $\mu = r$.
We know the solution to the Geometric Brownian motion

$$S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t + \sigma W_t^{\mathbb{P}}}$$

Let's consider a call option on the stock S with expiration at time T and strike K. The value of this option at time T is

$$V_T = \max(S_T - K, 0)$$

From the no-arbitrage pricing theory we have D_tV_t is a local martinage and under bounded regularity conditions we have

$$D_tV_t = \mathbb{E}^{\mathbb{P}}[D_TV_T|\mathcal{F}_t] \tag{1}$$

i.e. D_tV_t is a martingale.
Solving we have,

$$\begin{aligned} V_t &= \frac{1}{D_t}\mathbb{E}^{\mathbb{P}}[D_TV_T|\mathcal{F}_t] \\ &= e^{rt}\mathbb{E}^{\mathbb{P}}[e^{-rT}\max(S_T - K, 0)|\mathcal{F}_t] \\ &= e^{-r(T-t)}\mathbb{E}^{\mathbb{P}}[\max(S_te^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma W_{T-t}^{\mathbb{P}}} - K, 0)|\mathcal{F}_t] \\ &= e^{-r(T-t)}\int_{-\infty}^{\infty}\max(S_te^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}x} - K, 0)\frac{1}{\sqrt{2}}e^{-x^2}dx \end{aligned}$$

Denoting $\tau = T - t$ and $d_1 = \frac{1}{\sigma\sqrt{\tau}}\big[\log(\frac{K}{S_t}) - (r - \frac{1}{2}\sigma^2)\tau\big]$, we have

$$
\begin{aligned}
V_t &= e^{-r\tau} \int_{d_1}^{\infty} \big(S_t e^{(r-\frac{1}{2}\sigma^2)\tau+\sigma\sqrt{\tau}x} - S_t e^{(r-\frac{1}{2}\sigma^2)\tau+\sigma\sqrt{\tau}d_1}\big) \frac{1}{\sqrt{2}} e^{\frac{-x^2}{2}} dx \\
&= S_t e^{-\frac{1}{2}\sigma^2\tau} \int_{d_1}^{\infty} \big(e^{\sigma\sqrt{\tau}x} - e^{\sigma\sqrt{\tau}d_1}\big) \frac{1}{\sqrt{2}} e^{\frac{-x^2}{2}} dx \\
&= S_t e^{-\frac{1}{2}\sigma^2\tau} \big[\int_{-\infty}^{-d_1} e^{-\sigma\sqrt{\tau}x} \frac{1}{\sqrt{2}} e^{\frac{-x^2}{2}} dx - \int_{-\infty}^{-d_1} e^{\sigma\sqrt{\tau}d_1} \frac{1}{\sqrt{2}} e^{\frac{-x^2}{2}} dx \big] \\
&= S_t e^{-\frac{1}{2}\sigma^2\tau} \big[e^{\frac{1}{2}\sigma^2\tau} \int_{-\infty}^{-d_1} \frac{1}{\sqrt{2}} e^{\frac{-(x+\sigma\sqrt{\tau})^2}{2}} dx - e^{\sigma\sqrt{\tau}d_1} \int_{-\infty}^{-d_1} \frac{1}{\sqrt{2}} e^{\frac{-x^2}{2}} dx \big] \\
&= S_t \int_{-\infty}^{-d_1+\sigma\sqrt{\tau}} \frac{1}{\sqrt{2}} e^{-y^2} dy - K e^{(-r-\frac{1}{2}\sigma^2)\tau} \int_{-\infty}^{-d_1} \frac{1}{\sqrt{2}} e^{\frac{-x^2}{2}} dx \\
&= S_t N(-d_1 + \sigma\sqrt{\tau}) - K e^{-r\tau} N(-d_1)
\end{aligned}
$$

where N is the cumulative distribution function of the standard normal distribution.