Numeraires

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Abstract

In these notes we introduce numeraires and theorems related to change of numeraires along with their applications.

1 Change of Measure

Definition 1 (Radon-Nikodym Derivative)**.** *Consider two equivalent probability measures $\mathbb{P}$ and $\hat{\mathbb{P}}$ on a measurable space (Ω, Σ). The Radon-Nikodym derivate $d\hat{\mathbb{P}}/d\mathbb{P} : \Omega \to \mathbb{R}$ is defined such that for any subset $A, \Omega \supseteqq A \in \Sigma$,*

$$\int_A d\hat{\mathbb{P}} = \int_A (d\hat{\mathbb{P}}/d\mathbb{P}) d\mathbb{P}. \tag{1}$$

Suppose $(\Omega, \mathcal{F}, \mathcal{F}_t)$ is a filtered probability space, then note from the above definition we have

$$(d\hat{\mathbb{P}}/d\mathbb{P})|_{\mathcal{F}_t} = \mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})|\mathcal{F}_t] \tag{2}$$

$(d\hat{\mathbb{P}}/d\mathbb{P})|_{\mathcal{F}_t}$ is also written as $(d\hat{\mathbb{P}}/d\mathbb{P})_t$ and is thus a martingale stochastic process (by iterated conditioning) in $\mathbb{P}$.

Example 1 Let's consider a simple example to better understand Radom Nikodym derivative. Consider a die roll. We can assign mulitple probability distributions to the outcomes.

ω	$\mathbb{P}$	$\hat{\mathbb{P}}$	$d\hat{\mathbb{P}}/d\mathbb{P}$
1	1/6	1/2	3
2	1/6	1/4	3/2
3	1/6	1/8	3/4
4	1/6	1/16	3/8
5	1/6	1/32	3/16
6	1/6	1/32	3/16

The probability in $\hat{\mathbb{P}}$ of getting an odd number is $\int_{\omega \in \{1,3,5\}} d\hat{\mathbb{P}} = 1/2 + 1/8 + 1/32 = 21/32 = 3 * 1/6 + 3/4 * 1/6 + 3/16 * 1/6 = \int_{\omega \in \{1,3,5\}} (d\hat{\mathbb{P}}/d\mathbb{P}) d\mathbb{P}$

Theorem 2 (Abstract Bayes' Theorem)**.** *Let $\mathbb{P}$ and $\hat{\mathbb{P}}$ be two measures on measurable space $(\Omega, \mathcal{F})$. Let $\mathcal{G} \subset \mathcal{F}$ be another sigma algebra on Ω. Then for any $A \in G$ and random variable X*

$$\mathbb{E}^{\hat{\mathbb{P}}}[X|G] = \frac{\mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})X|G]}{\mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})|G]}. \tag{3}$$

Proof. We show that for any $A \in G$

$$\mathbb{E}^{\hat{\mathbb{P}}}[X|G]\mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})|G] = \mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})X|G]$$

Since the random variables involved are constant over A, we can check equality on integrals over A.

$$\begin{aligned}
\int_A \mathbb{E}^{\hat{\mathbb{P}}}[X|G]\mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})|G]d\mathbb{P} &= \int_A \mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})\mathbb{E}^{\hat{\mathbb{P}}}[X|G]|G]d\mathbb{P} && (\mathbb{E}^{\hat{\mathbb{P}}}[X|G] \text{ is G measurable}) \\
&= \int_A (d\hat{\mathbb{P}}/d\mathbb{P})\mathbb{E}^{\hat{\mathbb{P}}}[X|G]d\mathbb{P} && \text{(definition of conditional expectation)} \\
&= \int_A \mathbb{E}^{\hat{\mathbb{P}}}[X|G]d\hat{\mathbb{P}} && \text{(definition of Radon Nikodym derivative)} \\
&= \int_A X d\hat{\mathbb{P}} && \text{(definition of conditional expectation)} \\
&= \int_A (d\hat{\mathbb{P}}/d\mathbb{P})X d\mathbb{P} && \text{(definition of Radon Nikodym derivative)} \\
&= \int_A \mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})X|G]d\mathbb{P} && \text{(definition of conditional expectation).}
\end{aligned}$$

□

Remark. *Taking $X = V_T$ where V_s is $\mathcal{F}_s$ adapted and $G = \mathcal{F}_t$ for $t < T$ in the above theorem, we get the very useful formula for measure change for conditional expectations on filtered spaces,*

$$\mathbb{E}^{\hat{\mathbb{P}}}[V_T|\mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}[\frac{(d\hat{\mathbb{P}}/d\mathbb{P})_T}{(d\hat{\mathbb{P}}/d\mathbb{P})_t}V_T|\mathcal{F}_t] \tag{4}$$

Proof.

$$\begin{aligned}
\mathbb{E}^{\hat{\mathbb{P}}}[V_T|\mathcal{F}_t] &= \frac{\mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})V_T|\mathcal{F}_t]}{\mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})|\mathcal{F}_t]} && \text{(abstract Bayes' theorem)} \\
&= \frac{\mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})V_T|\mathcal{F}_T]|\mathcal{F}_t]}{\mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})|\mathcal{F}_t]} && \text{(iterated conditioning)} \\
&= \frac{\mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})|\mathcal{F}_T]V_T|\mathcal{F}_t]}{\mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})|\mathcal{F}_t]} && (V_T \text{ is } \mathcal{F}_T \text{ measureable}) \\
&= \frac{\mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})_T V_T|\mathcal{F}_t]}{(d\hat{\mathbb{P}}/d\mathbb{P})_t} && \text{(martigale property of Radon Nikodym derivative)} \\
&= \mathbb{E}^{\mathbb{P}}[\frac{(d\hat{\mathbb{P}}/d\mathbb{P})_T}{(d\hat{\mathbb{P}}/d\mathbb{P})_t}V_T|\mathcal{F}_t] && ((d\hat{\mathbb{P}}/d\mathbb{P})_t \text{ is } \mathcal{F}_t \text{ measurable}).
\end{aligned}$$

□

We consider processes until terminal time S i.e. $\mathcal{F} = \mathcal{F}_S$ and $(d\hat{\mathbb{P}}/\mathbb{P}) = (d\hat{\mathbb{P}}/\mathbb{P})_S$. We take a strictly positive martingale process to be the Random Nikodym derivative:

$$df_t = f_t\sigma(t)dW_t; \;\; f_t = e^{-\int_0^t \frac{1}{2}\sigma^2(s)ds + \int_0^t \sigma(s)dW_s}. \tag{5}$$

Theorem 3 (Girsanov Theorem)**.** *If $W_{1,t}$ is a Brownian motion in $\mathbb{P}$ and the Radon-Nikodym derivative $(d\hat{\mathbb{P}}/d\mathbb{P})_t = f_t$ is given by $f_t = e^{-\int_0^t \frac{1}{2}\sigma^2(s)ds + \int_0^t \sigma(s)dW^{\mathbb{P}}_{2,s}}$ then $W_t - \int_0^t \sigma(s)dW_{1,s}dW_{2,s}$ is a Brownian motion in $\hat{\mathbb{P}}$.*

Proof. We show that $X_t = W_{1,t} - \int_0^t \sigma(s)dW_{1,s}dW_{2,s}$ follows normal distribution with mean 0 and variance t in $\hat{\mathbb{P}}$. Other required properties can be verified easily. We show that the moment generating function of

X_t is same as that of normal distribution with mean 0 and variance t.

$$\begin{aligned}
\mathbb{E}^{\hat{\mathbb{P}}}[e^{-yX_t}] &= \mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})_t e^{-yX_t}] \\
&= \mathbb{E}^{\mathbb{P}}[e^{-\int_0^t \frac{1}{2}\sigma^2(s)ds + \int_0^t \sigma(s)dW_{2,s}} e^{-yX_t}] \\
&= e^{-\int_0^t \frac{1}{2}\sigma^2(s)ds}\mathbb{E}^{\mathbb{P}}[e^{\int_0^t \sigma(s)dW_{2,s} - yX_t}] \\
&= e^{-\int_0^t \frac{1}{2}\sigma^2(s)ds}\mathbb{E}^{\mathbb{P}}[e^{\int_0^t \sigma(s)dW_{2,s} - y(W_{1,t} - \int_0^t \sigma(s)dW_{1,s}dW_{2,s})}] \\
&= e^{-\int_0^t \frac{1}{2}\sigma^2(s)ds}\mathbb{E}^{\mathbb{P}}[e^{\int_0^t (\sigma(s)dW_{2,s} - ydW_{1,s} + y\sigma(s)dW_{1,s}dW_{2,s})}].
\end{aligned}$$

Define the martingale $Z_{y,t} = \int_0^t (\sigma(s)dW_{2,s} - ydW_{1,s})$. Then $e^{Z_{y,t} - \frac{1}{2}\int_0^t dZ_{y,s}dZ_{y,s}}$ is a martingale as well, with

$$dZ_{y,s}dZ_{y,s} = (\sigma^2(s) + y^2)ds - 2y\sigma dW_{1,s}dW_{2,s}.$$

We then have

$$\begin{aligned}
\mathbb{E}^{\hat{\mathbb{P}}}[e^{-yX_t}] &= e^{-\int_0^t \frac{1}{2}\sigma^2(s)ds}\mathbb{E}^{\mathbb{P}}[e^{Z_{y,t} - \frac{1}{2}\int_0^t dZ_{y,s}dZ_{y,s} + \frac{1}{2}\int_0^t (\sigma^2(s)+y^2)ds}] \\
&= e^{\frac{1}{2}y^2 t}\mathbb{E}^{\mathbb{P}}[e^{Z_{y,t} - \frac{1}{2}\int_0^t dZ_{y,s}dZ_{y,s}}] \\
&= e^{\frac{1}{2}y^2 t}(e^{Z_{y,t} - \frac{1}{2}\int_0^t dZ_{y,s}dZ_{y,s}})|_{t=0} \\
&= e^{\frac{1}{2}y^2 t}.
\end{aligned}$$

□

2 Numeraires

Definition 4 (Numeraire)**.** *A Numeraire is a strictly positive price process of a tradable relative to which prices of all other tradables are expressed.*

A savings account which earns interest at the instantaneous interest rate can be taken as a numeraire. The value of the savings account at any point is given by

$$A_t = e^{\int_0^t r(t)dt} = 1/D(t) \tag{6}$$

where $D(t)$ is the discount factor.

The fact that discounted trade prices are martingales in risk-neutral measure can then also be stated as: tradable prices in savings account (or money market) numeraire are martingales.

Consider another numeraire N_t. Since the numeraire is itself a price process, $D_t N_t$ is a martingale and we can take it as a Radon-Nikodym derivative, with a suitable normalization so that $\mathbb{E}^{\mathbb{P}}[(d\mathbb{N}/d\mathbb{P})] = 1$, giving

$$\mathbb{E}^{\mathbb{N}}[\frac{X_T}{N_T}|\mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}[\frac{(d\mathbb{N}/d\mathbb{P})_T}{(d\mathbb{N}/d\mathbb{P})_t}\frac{X_T}{N_T}|\mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}[\frac{D_T N_T}{D_t N_t}\frac{X_T}{N_T}|\mathcal{F}_t] = \frac{1}{D_t N_t}\mathbb{E}^{\mathbb{P}}[D_T X_T|\mathcal{F}_t] = \frac{X_t}{N_t} \tag{7}$$

where $\mathbb{P}$ is the risk neutral measure and $\mathbb{N}$ is the measure corresponding to the numeraire N_t.

Remark. *The above equation shows that tradable prices with respect to a numeraire are martingales in the measure corresponding to the numeraire.*

Risk neutral measure corresponds to the savings account (or money market) numeraire.

Definition 5 (T-forward measure)**.** *If we take N_t to be the price of a riskless bond maturing at time T, the corresponding measure as the T-forward measure.*

$$X_t/B(t,T) = \mathbb{E}^T[X_T/B(T,T)] = \mathbb{E}^T[X_T] \tag{8}$$

We see that that expectation of X_T in the T-forward measure gives the forward price $X_t/B(t,T)$ of the trade with expiration date T, hence the name T-forward measure. From the above equation we also notice that expiration T forward price process on an asset is martingale in T-forward measure.

Example 2 Option Pricing To see how measure changes can be used in pricing, let's take the example of Black Scholes option pricing. For an option on stock with expiry at T and strike K, we have the valuation forumula

$$V_t = \frac{1}{D_t}\mathbb{E}^{\mathbb{P}}[D_T \max(S_T - K, 0)|\mathcal{F}_t]$$

where $\mathbb{P}$ is the risk neutral probability measure, D_t is the discount factor and S_t is the stock price which follows the Black Scholes diffusion

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t$$

where r is the riskfree interest rate and W_t is a Brownian motion in $\mathbb{P}$.

Denote by $\mathbb{I}_{S_T>K}$ the indicator variable which takes value 1 is $S_T > K$ and 0 otherwise. We then have

$$\begin{aligned}
V_t &= \frac{1}{D_t}\mathbb{E}^{\mathbb{P}}[D_T \mathbb{I}_{S_T>K}(S_T - K)|\mathcal{F}_t] \\
&= \frac{1}{D_t}\mathbb{E}^{\mathbb{P}}[D_T \mathbb{I}_{S_T>K} S_T|\mathcal{F}_t] - \frac{1}{D_t}\mathbb{E}^{\mathbb{P}}[D_T \mathbb{I}_{S_T>K} K|\mathcal{F}_t]
\end{aligned}$$

Taking $\mathbb{S}$ to be the measure corresponding to stock price as numeraire, and taking $\mathbb{T}$ to be the measure corresponding to T-expiry bond as numeraire, we have

$$\begin{aligned}
V_t &= \frac{1}{D_t}\mathbb{E}^{\mathbb{S}}[D_T \frac{(d\mathbb{P}/d\mathbb{S})_T}{(d\mathbb{P}/d\mathbb{S})_t} \mathbb{I}_{S_T>K} S_T|\mathcal{F}_t] - \frac{1}{D_t}\mathbb{E}^{\mathbb{T}}[D_T \frac{(d\mathbb{P}/d\mathbb{T})_T}{(d\mathbb{P}/d\mathbb{T})_t} \mathbb{I}_{S_T>K} K|\mathcal{F}_t] \\
&= \frac{1}{D_t}\mathbb{E}^{\mathbb{S}}[D_T \frac{(d\mathbb{S}/d\mathbb{P})_t}{(d\mathbb{S}/d\mathbb{P})_T} \mathbb{I}_{S_T>K} S_T|\mathcal{F}_t] - \frac{1}{D_t}\mathbb{E}^{\mathbb{T}}[D_T \frac{(d\mathbb{T}/d\mathbb{P})_t}{(d\mathbb{T}/d\mathbb{P})_T} \mathbb{I}_{S_T>K} K|\mathcal{F}_t] \\
&= \frac{1}{D_t}\mathbb{E}^{\mathbb{S}}[D_T \frac{D_t S_t}{D_T S_T} \mathbb{I}_{S_T>K} S_T|\mathcal{F}_t] - \frac{1}{D_t}\mathbb{E}^{\mathbb{T}}[D_T \frac{D_t(B(t,T))}{D_T B(T,T)} \mathbb{I}_{S_T>K} K|\mathcal{F}_t] \\
&= S_t \mathbb{E}^{\mathbb{S}}[\mathbb{I}_{S_T>K}|\mathcal{F}_t] - K B(t,T)\mathbb{E}^{\mathbb{T}}[\mathbb{I}_{S_T>K}|\mathcal{F}_t]
\end{aligned}$$

where $B(t,T)$ is the bond price with the diffusion

$$dB(t,T)/B(t,T) = rdt + \sigma_2 dW_{2,t}$$

where $W_{2,t}$ is another Brownian motion in $\mathbb{P}$ such that $dW_t dW_{2,t} = \rho dt$.

Using Girsaonov's theorem, we have

$$\frac{dS_t}{S_t} = rdt + \sigma(dW_t^{\mathbb{S}} + \sigma dt) = rdt + \sigma(dW_t^{\mathbb{T}} + \sigma_2 \rho dt) \tag{9}$$

where $dW_t^{\mathbb{S}} = dW_t - \sigma dt$ is a Brownian motion in $\mathbb{S}$ and $dW_t^{\mathbb{T}} = dW_t^{\mathbb{S}} - \sigma_2 \rho dt$ is a Brownian motion in $\mathbb{T}$. Rearranging,

$$\frac{dS_t}{S_t} = (r + \sigma^2)dt + \sigma dW_t^{\mathbb{S}} = (r + \sigma\sigma_2\rho)dt + \sigma dW_t^{\mathbb{T}} \tag{10}$$

$$S_T = S_t e^{(r+\frac{1}{2}\sigma^2)\tau + \sigma W_\tau^{\mathbb{S}}} = S_t e^{(r+\sigma(\sigma_2\rho - \frac{1}{2}\sigma)\tau + \sigma W_\tau^{\mathbb{T}}} \tag{11}$$

where $\tau = T - t$.

$$\begin{aligned}
\mathbb{E}^{\mathbb{S}}[\mathbb{I}_{S_T>K}|\mathcal{F}_t] &= \mathbb{E}^{\mathbb{S}}[\mathbb{I}_{S_t e^{(r+\frac{1}{2}\sigma^2)\tau+\sigma W^{\mathbb{S}}_\tau}>K}|\mathcal{F}_t] \\
&= \mathbb{E}^{\mathbb{S}}[\mathbb{I}_{(r+\frac{1}{2}\sigma^2)\tau+\sigma W^{\mathbb{S}}_\tau>\log(K/S_t)}|\mathcal{F}_t] \\
&= \mathbb{E}^{\mathbb{S}}[\mathbb{I}_{\sigma W^{\mathbb{S}}_\tau>\log(K/S_t)-(r+\frac{1}{2}\sigma^2)\tau}|\mathcal{F}_t] \\
&= \mathbb{E}^{\mathbb{S}}[\mathbb{I}_{\frac{1}{\sqrt{\tau}}W^{\mathbb{S}}_\tau>\frac{1}{\sigma\sqrt{\tau}}(\log(K/S_t)-(r+\frac{1}{2}\sigma^2)\tau)}|\mathcal{F}_t] \\
&= N(-\frac{1}{\sigma\sqrt{\tau}}(\log(K/S_t)-(r+\frac{1}{2}\sigma^2)\tau)) \\
&= N(-d_1+\sigma\sqrt{\tau})
\end{aligned}$$

where N is the cumulative normal distribution, and $d_1 = \frac{1}{\sigma\sqrt{\tau}}(\log(K/S_t) - (r - \frac{1}{2}\sigma^2)\tau)$.

Similarly we have,

$$\begin{aligned}
\mathbb{E}^{\mathbb{T}}[\mathbb{I}_{S_T>K}|\mathcal{F}_t] &= \mathbb{E}^{\mathbb{T}}[\mathbb{I}_{S_t e^{(r+\sigma(\sigma_2\rho-\frac{1}{2}\sigma)\tau+\sigma W^{\mathbb{S}}_\tau}>K}|\mathcal{F}_t] \\
&= \mathbb{E}^{\mathbb{T}}[\mathbb{I}_{(r+\sigma(\sigma_2\rho-\frac{1}{2}\sigma)\tau+\sigma W^{\mathbb{S}}_\tau>\log(K/S_t)}|\mathcal{F}_t] \\
&= \mathbb{E}^{\mathbb{T}}[\mathbb{I}_{\sigma W^{\mathbb{S}}_\tau>\log(K/S_t)-(r+\sigma(\sigma_2\rho-\frac{1}{2}\sigma)\tau}|\mathcal{F}_t] \\
&= \mathbb{E}^{\mathbb{T}}[\mathbb{I}_{\frac{1}{\sqrt{\tau}}W^{\mathbb{S}}_\tau>\frac{1}{\sigma\sqrt{\tau}}(\log(K/S_t)-(r+\sigma(\sigma_2\rho-\frac{1}{2}\sigma)\tau)}|\mathcal{F}_t] \\
&= N(-\frac{1}{\sigma\sqrt{\tau}}(\log(K/S_t)-(r+\sigma(\sigma_2\rho-\frac{1}{2}\sigma)\tau)) \\
&= N(-d_1+\sigma_2\rho\sqrt{\tau}))
\end{aligned}$$

And finally we have,

$$V_t = S_t N(-d_1+\sigma\sqrt{\tau}) - KB(t,T)N(-d_1+\sigma_2\rho\sqrt{\tau})$$

Note that $\sigma d\tau$ and $\sigma_2\rho d\tau$ are the drift correction terms we get to stock price Brownian motion using Girsanov's theorem to change measure to stock and bond numeraires respectively.