Fixed Income

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Abstract

In these notes we study some of the simple interest rate models for pricing of fixed income instruments. We will consider single curve framework i.e. there is only one interest rate in the market.

1 Interest Rate Instruments

Some of the popular traded fixed income instruments are bonds, swaps, futures and forwards.

Definition 1 (Bond)**.** *A bond is an instrument which pays a specified notional amount at maturity and periodic coupons until then. A zero coupon bond ia a bond which does not pay coupons, only the notional at maturity.*

One of the major differences in modelling interest rates as opposed to equities is that the current interest rate is a function of the maturity of the bond, i.e. it is a time-indexed curve as opposed to a number.

Definition 2 (Instantaneous Forward Rate)**.** *The instantaneous forwad rate $f(t,T)$ is the rate charged for short term borrowing/lending at time T, and quoted at time t. If we enter into a contract at time t, to borrow money from time T to T+dT, we would be charged an interest of $f(t,T)dT$.*

Remark. *The above definition of instantaneous forward rate implies the following relationship with prices of zero-coupon bonds with maturity at time T and notional 1, henceforth denoted as $P(t,T)$:*

$$P(t,T) = e^{-\int_t^T (t,s)ds}$$

which can be seen by using compounding formula over short intervals from t to T and then taking the limit as intervals approach length 0.

The initial instantaneous forward rate curve $f(0,T)$ can be inferred from market bond prices $P(0,T)$ and other fixed income instruments. However, the instruments are only available for certain tenors and the interpolation of curve to other points if one of the themes in rate curve modelling.

Definition 3 (Short Interest Rate)**.** *Short interest rate $r(t)$ is the rate charged for short term borrowing at time t, and quoted at time t.*

$$r(t) = f(t,t)$$

2 Heath-Jarrow-Morton Framework

Heath-Jarrow-Morton framework provides restrictions on modelling of the diffusions of instantaneous forward rates so that they are arbitrage-free.

The framework assumes that the price process of zero coupon bond of each maturity follows the following diffusion in risk neutral measure:

$$dP(t,T) = r(t)B(t,T)dt + \nu(t,T)P(t,T)dW_t$$

where $\nu_(t,T)$ can be any $\mathcal{F}_t$ adapted process.

Note that we have already imposed the no-arbitrage condition on bond prices by setting the drift term to be $r(t)P(t,T)$. $v(t,T)$ specifies a curve of volatility curve indexed with respect to the maturity of the bond T and is called the volatility term structure.

Now let's see what this form of bond diffusion implies for the instantaneous forward rate.

$$\begin{aligned}
f(t,T) &= -\frac{\partial}{\partial T}[\ln(P(t,T))] \\
d\ln(P(t,T)) &= (r(t) - \frac{1}{2}\nu^2(t,T))dt + \nu(t,T)dW_t && \text{(Using Ito's Lemma)} \\
df(t,T) &= -\frac{\partial}{\partial T}[d\ln(P(t,T))] \\
&= -\frac{\partial}{\partial T}[(r(t) - \frac{1}{2}\nu^2(t,T))dt + \nu(t,T)dW_t] \\
&= \frac{\partial}{\partial T}[\frac{1}{2}\nu^2(t,T))]dt - \frac{\partial}{\partial T}[\nu(t,T)]dW_t \\
&= \nu(t,T)\nu_T(t,T)dt - \nu(t,T)dW_t && (\nu_T(t,T) \equiv \frac{\partial}{\partial T}\nu(t,T)) \\
&= \sigma(t,T)\bigg(\int_t^T \sigma(t,s)ds\bigg)dt - \sigma(t,T)dW_t && (\sigma(t,T) \equiv \nu_T(t,T), P(t,t) = 1 \Rightarrow \nu(t,t) = 0) \\
&= \sigma(t,T)\bigg(\int_t^T \sigma(t,s)ds\bigg)dt + \sigma(t,T)dW_t^* && \text{(flip Brownian motion)}
\end{aligned}$$

We have therefore derived the drift of the instantaneous forward rate curve in terms of the volatility term structure.

If $\sigma(t,T)$ is deterministic then the instantaneous forward rates have Gaussian distribution, and the bond prices have log-normal distribution. The models in this case are called Gaussian HJM models.

Note that we have used a single Brownian function to generate diffusion of the entire instantaneous forward rate curve. This implies, in Gaussian case, that different points on instantaneous forward curve move in same direction in proportion to their volatility, on action of Brownian motion, and hence are perfectly correlated. This correlation can be relaxed by using multiple Brownian motions with different volatility term structure coeffecients.

3 Hull-White Model

Consider one-factor HJM diffusion. Instrument pricing is much more tractable if we restrict the volatility term-structure to be deterministic, separable and of the form:

$$\sigma(t,T) = \sigma(t)e(-\int_t^T \alpha(s)ds))$$

Specifically, in Hull-White model we have time-independent $\alpha(s) = \alpha$ and $\sigma(t) = \sigma$.

In this case,

$$\begin{aligned}
df(t,T) =& \sigma e^{-\alpha(T-t)}\bigg(\int_t^T \sigma e^{-\alpha(u-t)}du\bigg)dt + \sigma e^{-\alpha(T-t)}dW_t \\
=& \sigma^2 e^{-\alpha(T-t)}\bigg(\int_t^T e^{-\alpha(u-t)}du\bigg)dt + \sigma e^{-\alpha(T-t)}dW_t \\
=& \frac{\sigma^2}{\alpha}e^{-\alpha(T-t)}\Big(1 - e^{-\alpha(T-t)}\Big)dt + \sigma e^{-\alpha(T-t)}dW_t
\end{aligned}$$

$$\begin{aligned} f(t,T) =& f(0,T) + \int_0^t \frac{\sigma^2}{\alpha} e^{-\alpha(T-u)} \Big(1 - e^{-\alpha(T-u)}\Big) du + \int_0^t \sigma e^{-\alpha(T-u)} dW_u \\ =& f(0,T) + \frac{\sigma^2}{\alpha} \int_0^t \Big(e^{-\alpha(T-u)} - e^{-2\alpha(T-u)}\Big) du + \sigma \int_0^t e^{-\alpha(T-u)} dW_u \\ =& f(0,T) + \frac{\sigma^2}{\alpha^2} \Big(e^{-\alpha(T-t)} - e^{-\alpha T} - \frac{1}{2}(e^{-2\alpha(T-t)} - e^{-2\alpha T})\Big) + \sigma \int_0^t e^{-\alpha(T-u)} dW_u \end{aligned}$$

Using Ito isometry, $f(t,T)$ therefore is normally distributed as

$$f(t,T) \sim N\Big(f(0,T) + \frac{\sigma^2}{\alpha^2}(\frac{e^{\alpha t}-1}{e^{\alpha T}} - \frac{e^{2\alpha t}-1}{2e^{2\alpha T}}), \frac{\sigma^2}{2\alpha}\frac{e^{2\alpha t}-1}{e^{2\alpha T}}\Big)$$

where $N(\mu, \sigma^2)$ is a normal distribution with mean μ and variance σ^2.
We then have

$$\begin{aligned} \int_t^T f(t,s)ds =& \int_t^T \Big(f(0,s) + \frac{\sigma^2}{\alpha^2}(\frac{e^{\alpha t}-1}{e^{\alpha s}} - \frac{e^{2\alpha t}-1}{2e^{2\alpha s}})\Big) ds + \sigma \int_t^T \int_0^t e^{-\alpha(s-u)} dW_u ds \\ =& \int_t^T f(0,s)ds + \frac{\sigma^2}{\alpha^3}\big((e^{\alpha t}-1)(e^{-\alpha t} - e^{-\alpha T}) - \frac{1}{4}(e^{2\alpha t}-1)(e^{-2\alpha t} - e^{-2\alpha T})\big) + \sigma \int_t^T \int_0^t e^{-\alpha(s-u)} dW_u ds \\ =& \int_t^T f(0,s)ds + \frac{\sigma^2}{\alpha^3}\big((1-e^{-\alpha t})(1-e^{-\alpha(T-t)}) - \frac{1}{4}(1-e^{-2\alpha t})(1-e^{-2\alpha(T-t)})\big) + \sigma \int_t^T \int_0^t e^{-\alpha(s-u)} dW_u ds \\ =& \int_t^T f(0,s)ds + \frac{\sigma^2}{\alpha^3}\big((1-e^{-\alpha t})(1-e^{-\alpha(T-t)}) - \frac{1}{4}(1-e^{-2\alpha t})(1-e^{-2\alpha(T-t)})\big) \\ &+ \frac{\sigma}{\alpha}(e^{-\alpha t} - e^{-\alpha T}) \int_0^t e^{\alpha u} dW_u \end{aligned}$$

Therefore,

$$\int_t^T f(t,s)ds \sim N\Big(\int_t^T f(0,s)ds + \frac{\sigma^2}{\alpha^3}\big((1-e^{-\alpha t})(1-e^{-\alpha(T-t)}) - \frac{1}{4}(1-e^{-2\alpha t})(1-e^{-2\alpha(T-t)})\big), \frac{\sigma^2}{2\alpha^3}(1-e^{-\alpha(T-t)})^2(1-e^{-2\alpha t})\Big) \tag{1}$$

and the bond price

$$P(t,T) = e^{-\int_t^T f(t,s)ds}$$

is log-normally distributed.

The Hull-White model can be specified completely in terms of the short rate diffusion as well, and is therefore called a short-rate model. Let us see how this re-parametrization works.

$$\begin{aligned} r(t) =& f(t,t) \\ =& f(0,t) + \frac{\sigma^2}{\alpha^2}\Big(1 - e^{-\alpha t} - \frac{1}{2}(1 - e^{-2\alpha t})\Big) + \sigma \int_0^t e^{-\alpha(t-u)} dW_u \\ =& f(0,t) + \frac{\sigma^2}{\alpha^2}\Big(1 - e^{-\alpha t} - \frac{1}{2}(1 - e^{-2\alpha t})\Big) + \sigma e^{-\alpha t} \int_0^t e^{\alpha u} dW_u \\ =& f(0,t) + \frac{\sigma^2}{2\alpha^2}\big(1 - e^{-\alpha t}\big)^2 + \sigma e^{-\alpha t} \int_0^t e^{\alpha u} dW_u \end{aligned}$$

Notice here that the stochastic evolution of both $\int_t^T f(t,s)ds$ and $r(t)$ is determined by the term $\int_0^t e^{\alpha u} dW_u$ and therefore the bond yields $\ln P(t,T)$ can be specified as a linear function of $r(t)$ with coeffecients of this linear relationship being deterministic functions of t and T. Models with such linear relationship between short rate and yield term structure are known as affine term structure models.

$$
\begin{aligned}
P(t,T) &= \exp\big(A_1(t,T) + B_1(t,T)\int_0^t e^{\alpha u} dW_u\big) \\
&= \exp\big(A_2(t,T) + B_2(t,T) f(t,T)\big) \\
&= \exp\big(A_3(t,T) + B_3(t,T) r(t)\big)
\end{aligned}
$$

where $A_i(t,T)$ and $B_i(t,T)$ are deterministic functions of t and T.

Getting back to short rate diffusion specification, we have

$$
\begin{aligned}
dr(t) &= \Big(\frac{\partial}{\partial t} f(0,t) + \frac{\sigma^2}{\alpha}(e^{-\alpha t} - e^{-2\alpha t}) - \sigma\alpha e^{-\alpha t}\int_0^t e^{\alpha u} dW_u\Big)dt + \sigma e^{-\alpha t} e^{\alpha t} dW_t \\
&= \Big(\frac{\partial}{\partial t} f(0,t) + \frac{\sigma^2}{\alpha}(e^{-\alpha t} - e^{-2\alpha t}) - \alpha\Big(r(t) - (f(0,t) + \frac{\sigma^2}{2\alpha^2}\big(1 - e^{-\alpha t}\big)^2)\Big)\Big)dt + \sigma dW_t \\
&= \Big(\frac{\partial}{\partial t} f(0,t) + \frac{\sigma^2}{\alpha}(e^{-\alpha t} - e^{-2\alpha t}) + \alpha(f(0,t) + \frac{\sigma^2}{2\alpha^2}\big(1 - e^{-\alpha t}\big)^2) - \alpha r(t)\Big)dt + \sigma dW_t \\
&= \big(\theta(t) - \alpha r(t)\big)dt + \sigma dW_t
\end{aligned}
$$

where we have introduced the parameter $\theta(t)$ which is specified of terms of other deterministic parameters α, σ and initial instantaneous forward rate curve $f(0,\cdot)$. From this specification, we have another interpretation of *alpha* as the mean-reversion strength of short rate.

4 Future-Forward Basis

In this section we will work through calculations to compute the difference between LIBOR futures and LIBOR forwards. LIBOR rates were recently decommisioned but the calculations are instructive and the logic can be transferred to pricing of futures and forwards on other benchmark rates.

Definition 4 (LIBOR Rate). *LIBOR rates are(were) daily published interbank borrowing/lending rates for specific periods. For example, a 6-month LIBOR rate published at time t is the interest rate charged on borrowing from t to t + 6 months.*

$$L(t,T,T+\delta) = \frac{1}{\delta}\big(\frac{P(t,T)}{P(t,T+\delta)} - 1\big)$$

where $L(t,T,T+\delta)$ is the Libor Rate at t for borrowing between times t and $t+\delta$, and the formula is just the formula for simple interest rate.

Definition 5 (Forward Rate Agreement). *A δ LIBOR forward rate agreement maturing at T is a contract that pays at T:*

$$\frac{\delta(L(T,T+\delta) - R)}{1 + \delta L(T,T+\delta)}$$

where R is the interest rate specified in the agreement and $(T,T+\delta)$ is the reference period for interest rate.

Forward rate FRA is the value of R for which the price of this agreement is 0.

$$
\begin{aligned}
0 =& P(0,T)\mathbb{E}^T[\frac{\delta(L(T,T+\delta)-\text{FRA}(0))}{1+\delta L(T,T+\delta)}] \\
=& P(0,T)\mathbb{E}^T[\delta(L(T,T+\delta)-\text{FRA}(0))P(T,T+\delta)] \\
=& P(0,T)\mathbb{E}^{T+\delta}[\delta(L(T,T+\delta)-\text{FRA}(0))P(T,T+\delta)\frac{P(T,T)P(0,T+\delta)}{P(0,T)P(T,T+\delta)}] \\
=& P(0,T+\delta)\mathbb{E}^{T+\delta}[\delta(L(T,T+\delta)-\text{FRA}(0))] \\
\text{FRA}(0) =& \mathbb{E}^{T+\delta}[L(T,T,T+\delta)]
\end{aligned}
$$

Notice that $L(t,T,T+\delta)$ is a ratio of two prices plus a constant, and scaled by a constant. Therefore it is martingale in measure with $P(t,T)$ as numeraire i.e. T-forward measure. Therefore

$$\text{FRA}(t) = \mathbb{E}_t^{T+\delta}[L(T,T,T+\delta)] = L(t,T,T+\delta)$$

is known as forward LIBOR rate.

Definition 6 (Eurodollar Futures)**.** *Eurodollar futures are daily margined contracts that at maturity T settle at value $1-L(T,T+\delta)$ which is published at T.*

Eurodollar futures are settled in market at value $1-L(T,T+\delta)$ but for ease of comparison we will take futures value at T $\text{Fut}(T)$ to be $L(T,T+\delta)$. We can get the true futures value from $1-\text{Fut}(0)$.

Futures contracts on LIBOR are different from forwards in two major ways.

1. Futures contracts are daily margined. That is, the buyer does not pay for the contract initially, except for initial margin which is usually much smaller than the futures price. Then the movement in futures prices are settled daily; if the futures price goes up, the buyer has cash put in his margin account and if the futures price goes down, the buyer has cash withdrawn from this account.

2. Futures contracts are settled at the maturity of futures contract, which is start of the LIBOR period T, as opposed to forwards which are settled at end of LIBOR period $T+\delta$.

The futures price $\text{Fut}(t)$ is such that price of future cashflows from daily margining is 0 i.e. for all t

$$
\begin{aligned}
0 =& \frac{1}{D_t}\mathbb{E}_t^{\mathbb{P}}\left[\int_t^T D_s\frac{\partial}{\partial s}\text{Fut}(s)ds\right] \\
0 =& \mathbb{E}_t^{\mathbb{P}}\left[\int_t^T D_s\frac{\partial}{\partial s}\text{Fut}(s)ds\right]
\end{aligned}
$$

where $\frac{\partial}{\partial s}\text{Fut}(s)ds$ is the cashflow from margin in the time interval $(s,s+ds)$. Differentiating with respect to t,

$$
\begin{aligned}
0 =& \mathbb{E}_t^{\mathbb{P}}\left[-D_t\frac{\partial}{\partial t}\text{Fut}(t)\right] \\
=& -D_t\mathbb{E}_t^{\mathbb{P}}\left[\frac{\partial}{\partial t}\text{Fut}(t)\right] \\
0 =& \mathbb{E}_t^{\mathbb{P}}\left[\frac{\partial}{\partial t}\text{Fut}(t)\right]
\end{aligned}
$$

Integrating and using iterated condititioning, we get

$$\begin{aligned}\text{Fut}(0) =&\mathbb{E}_0^{\mathbb{P}}[\text{Fut}(T)]\\ =&\mathbb{E}_0^{\mathbb{P}}[L(T,T,T+\delta)]\end{aligned}$$

which says that the current futures price is the expected settlement price of the future in risk neutral measure.

The future-forward basis is largely sensitive to the covariance between rate fixing and money market discount factor.

$$\begin{aligned}\text{Fut}(0)-\text{FRA}(0) =&\mathbb{E}^{\mathbb{P}}[L(T,T,T+\delta)]-\mathbb{E}^{T+\delta}[L(T,T,T+\delta)]\\ =&\mathbb{E}^{\mathbb{P}}[L(T,T,T+\delta)]-\mathbb{E}^{\mathbb{P}}[L(T,T,T+\delta)\frac{P(T,T+\delta)D_T}{P(0,T+\delta)D_0}]\\ =&\frac{P(0,T+\delta)\mathbb{E}^{\mathbb{P}}[L(T,T,T+\delta)]-\mathbb{E}^{\mathbb{P}}[L(T,T,T+\delta)P(T,T+\delta)D_T]}{P(0,T+\delta)}\\ =&\frac{\mathbb{E}^{\mathbb{P}}[P(T,T+\delta)D_T]\mathbb{E}^{\mathbb{P}}[L(T,T,T+\delta)]-\mathbb{E}^{\mathbb{P}}[L(T,T,T+\delta)P(T,T+\delta)D_T]}{P(0,T+\delta)}\\ =&\frac{\text{Cov}^{\mathbb{P}}(P(T,T+\delta)D_T,L(T,T,T+\delta))}{P(0,T+\delta)}\\ =&\frac{\text{Cov}^{\mathbb{P}}(P(T,T+\delta)D_T,\frac{1}{P(T,T+\delta)}))}{\delta P(0,T+\delta)}\end{aligned}$$

where $\text{Cov}^{\mathbb{P}}(X,Y)$ is the covariance of X and Y in risk neutral measure.

We now compute the basis in Hull-White model.

$$\begin{aligned}\text{Fut}(0)-\text{FRA}(0) =&\mathbb{E}^{\mathbb{P}}[L(T,T,T+\delta)]-\mathbb{E}^{T+\delta}[L(T,T,T+\delta)]\\ =&\mathbb{E}^{\mathbb{P}}[L(T,T,T+\delta)]-L(0,T,T+\delta)\\ =&\mathbb{E}^{\mathbb{P}}[\frac{1}{\delta}(\frac{1}{P(T,T+\delta)}-1)]-\frac{1}{\delta}(\frac{P(0,T)}{P(0,T+\delta)}-1)\\ =&\frac{1}{\delta}\left(\mathbb{E}^{\mathbb{P}}[\frac{1}{P(T,T+\delta)}]-\frac{P(0,T)}{P(0,T+\delta)}\right)\end{aligned}$$

We therefore only need to compute

$$\mathbb{E}^{\mathbb{P}}[\frac{1}{P(T,T+\delta)}]=\mathbb{E}^{\mathbb{P}}[e^{\int_T^{T+\delta}f(T,s)ds}]$$

We know that $\int_T^{T+\delta}f(T,s)ds$ is normally distributed, say with mean μ and variance V which were derived earlier in expression 1.

$$\mathbb{E}^{\mathbb{P}}[e^{\int_T^{T+\delta}f(T,s)ds}]=e^{\mu+\frac{V}{2}}$$

where we used the fact that $E[e^{\sigma W_t}]=E[e^{\frac{\sigma^2 t}{2}}]$ that is the expecation of exponential of normal process with mean 0 is exponential of half of its variance. This can be verified by taking a log normal martingale process $dX_t/X_t=\sigma dW_t$ and using the property $E[X_T]=X_0$.

Rest of the algebra is straightforward and we leave that to the reader.

5 CMS Payoffs

Definition 7 (Swap)**.** *A receiver fixed-for-float swap is an instrument which pays a periodic floating interest rate (float leg) and receives a fixed interest rate specified in the swap contract (fixed leg), usually both on the same notional. The periodicity of the two legs can be different. The floating rate is usually determined from interbank borrowing rates (eg. LIBOR) or some other measure of prevalent borrowing costs in the market (eg. SOFR).*

From here on, we assume the notional to 1. The prices can be simply be scaled by notional to incorporate the notional.

Let $\tau^r_{i>0,i\leq N_r}$ and $\tau^p_{i>0,i\leq N_p}$ be the payment dates of receiver (fixed) leg and payer (floating) leg respectively with τ^r_0 and τ^p_0 being the swap start date. $\delta^r = \tau^r_i - \tau^r_{i-1}$ is the periodicity of the receiver leg and $\delta^p = \tau^p_i - \tau^p_{i-1}$ is the periodicity of the payer leg. The specified rate of the fixed leg is denoted as R.

The swap holder receives $\delta^r R$ at $\tau^r_{i>0}$ and pays $\delta^p L(\tau^p_{i-1}, \tau^p_{i-1}, \tau^p_i)$ at $\tau^p_{i>0}$.

Price of float leg is

$$\begin{aligned}
\text{Float}(0) =& \Sigma_{i>0}^{N_p} P(0,\tau^p_i)\mathbb{E}^{\tau^p_i}[\delta^p L(\tau^p_{i-1},\tau^p_{i-1},\tau^p_i)] \\
=& \Sigma_{i>0}^{N_p} P(0,\tau^p_i)\delta^p L(0,\tau^p_{i-1},\tau^p_i) \\
=& \Sigma_{i>0}^{N_p} P(0,\tau^p_i)(\frac{P(0,\tau^p_{i-1})}{P(0,\tau^p_i)} - 1) \\
=& \Sigma_{i>0}^{N_p} (P(0,\tau^p_{i-1}) - P(0,\tau^p_i)) \\
=& P(0,\tau^p_{N_p}) - P(0,\tau^p_0)
\end{aligned}$$

Definition 8 (Annuity)**.** *We define annuity Ann as the price of periodic payments of one dollar at the fixed leg payment dates $\tau^r_{i>0,i\leq N_r}$.*

$$\text{Ann}(0) = \Sigma_{i>0}^{N_r} P(0,\tau^r_i)\delta^r$$

Price of fixed leg is then

$$\begin{aligned}
\text{Fixed}(0) =& \Sigma_{i>0}^{N_r} P(0,\tau^r_i)\mathbb{E}^{\tau^r_i}[\delta^r R] \\
=& R\Sigma_{i>0}^{N_r} P(0,\tau^r_i)\delta^r \\
=& R\text{Ann}(0)
\end{aligned}$$

Definition 9 (Swap Rate)**.** *Swap rate is the fixed rate of the swap for which the price of the swap is 0.*

Let S be the swap rate, then

$$\begin{aligned}
S(t)\text{Ann}(t) =& P(t,\tau^p_{N_p}) - P(t,\tau^p_0) \\
S(t) =& (P(t,\tau^p_{N_p}) - P(t,\tau^p_0))/\text{Ann}(t)
\end{aligned}$$

Since annuity is itself a price, swap rate is the ratio of two prices and hence is martingale in the measure with annuity as numeraire.

$$S(t) = \mathbb{E}^{\text{Ann}}_t[S(T)]$$

Price of receiver swap can be now be re-written as

$$\begin{aligned}\text{Fixed}(0) - \text{Float}(0) =& R\text{Ann}(0) - S(0)\text{Ann}(0)\\ =& \text{Ann}(0)(R - S(0))\end{aligned}$$

Similarly, price of payer swap is

$$\text{Float}(0) - \text{Fixed}(0) = \text{Ann}(0)(S(0) - R)$$

Definition 10 (Swaption)**.** *A swaption is an option to enter a swap with swaption strike as the swap fixed rate. A call option is an option to enter a payer swap and a put option is an option to enter a receiver swap. The swap start date is the option exercise date.*

Price of call swaption with exercise date T and strike K is therefore

$$\begin{aligned}\text{Swo}(0) =& P(0,T)\mathbb{E}^T[\max(\text{Ann}(T)(S(T) - K), 0)]\\ =& P(0,T)\mathbb{E}^{\text{Ann}}[\max(\text{Ann}(T)(S(T) - K), 0)\frac{\text{Ann}(0)P(T,T)}{\text{Ann}(T)P(0,T)}]\\ =& \text{Ann}(0)\mathbb{E}^{\text{Ann}}[\max(S(T) - K, 0)]\\ =& \text{Ann}(0)\mathbb{E}^{\text{Ann}}[(S(T) - K)^+]\end{aligned}$$

Definition 11 (CMS Linked Products)**.** *A CMS (Constant Maturity Swap) linked product is a contract that references swap rates of swaps of a fixed maturity in their cashflows. For example, a CMS linked strip could pay at every 6 months the swap rate of a swap starting then and ending 10 years from then.*

Let us consider an CMS caplet referencing a swap rate S, maturing at time T, and with strike K. This product is cash settled at T and hence the price is

$$\text{CMSCap}(0) = P(0,T)\mathbb{E}^T[(S(T) - K)^+]$$

Comparing with a swaption, whereas the swaption pays the difference between swap rate and the fixed rate over the life of the swap, the CMS cap settles the difference in cash at the option expiry.

We could in principle calibrate a Hull-White model paremeters to market prices of bonds, futures, forwards and swaptions and then use the calibrated model to price the CMS caplet. But let us consider an alternative approach.

Consider a CMS caplet that settles at $T_p \geq T$ which allows for a payment delay.

$$\begin{aligned}\text{CMSCap}_{T_p}(0) =& P(0,T_p)\mathbb{E}^{T_p}[(S(T) - K)^+]\\ =& P(0,T_p)\mathbb{E}^{\text{Ann}}[(S(T) - K)^+\frac{P(T,T_p)\text{Ann}(0)}{P(0,T_p)\text{Ann}(T)}]\\ =& \text{Ann}(0)\mathbb{E}^{\text{Ann}}[(S(T) - K)^+\frac{P(T,T_p)}{\text{Ann}(T)}]\\ =& \text{Ann}(0)\mathbb{E}^{\text{Ann}}[(S(T) - K)^+\mathbb{E}^{\text{Ann}}[\frac{P(T,T_p)}{\text{Ann}(T)}|S(T) = s]]\end{aligned}$$

$M(s, T_p) = \mathbb{E}^{\text{Ann}}[\frac{P(T,T_p)}{\text{Ann}(T)}|S(T) = s]$ is called the Annuity Mapping function, and is generally linear in s and T_p in applicable domain.

$$\begin{aligned}M(s, T_p) =& \mathbb{E}^{\text{Ann}}[\frac{P(T,T_p)}{\text{Ann}(T)}|S(T) = s]\\ =& a(T_p)s + b(T_p)\end{aligned}$$

Once we have a distribution for $S(T)$ in annuity measure which can be inferred using swaption replication (looking at market values of swpations with same expiry and swap end date and various strikes), and the linear functions a and b, CMSCap price is a call on quadratic function of $S(T)$. We do not need a term-structure model like Hull-White to price this.

We derive of a and b using arbitrage conditions.

Condition 1. Martingale property.

$$\mathbb{E}^{\text{Ann}}[\mathbb{E}^{\text{Ann}}[\frac{P(T,T_p)}{\text{Ann}(T)}|S(T)=s]] = \frac{P(0,T_p)}{\text{Ann}(0)}$$

This implies

$$\begin{aligned}
\mathbb{E}^{\text{Ann}}[M(S(T),T_p)] =& \frac{P(0,T_p)}{\text{Ann}(0)} \\
\mathbb{E}^{\text{Ann}}[a(T_p)S(T)+b(T_p)] =& \frac{P(0,T_p)}{\text{Ann}(0)} \\
a(T_p)\mathbb{E}^{\text{Ann}}[S(T)]+b(T_p) =& \frac{P(0,T_p)}{\text{Ann}(0)} \\
a(T_p)S(0)+b(T_p) =& \frac{P(0,T_p)}{\text{Ann}(0)} \\
b(T_p) =& \frac{P(0,T_p)}{\text{Ann}(0)} - a(T_p)S(0) \\
M(s,T_p) =& a(T_p)(s-S(0)) + \frac{P(0,T_p)}{\text{Ann}(0)}
\end{aligned}$$

Condition 2. Definition of annuity in terms of zero coupon bonds.

$$\Sigma_{i=1}^{N_p}\delta_i^p P(T,\tau_i^p) = \text{Ann}(T)$$

where fixed leg is the payer leg. This implies

$$\begin{aligned}
\Sigma_{i=1}^{N_p}\delta_i^p \frac{P(T,\tau_i^p)}{\text{Ann}(T)} =& 1 \\
\Sigma_{i=1}^{N_p}\delta_i^p \mathbb{E}^{\text{Ann}}[\frac{P(T,\tau_i^p)}{\text{Ann}(T)}|S(T)=s] =& 1 \\
\Sigma_{i=1}^{N_p}\delta_i^p M(s,\tau_i^p) =& 1 \\
\Sigma_{i=1}^{N_p}\delta_i^p a(\tau_i^p)(s-S(0)) + \Sigma_{i=1}^{N_p}\delta_i^p \frac{P(0,\tau_i^p)}{\text{Ann}(0)} =& 1 \\
\Sigma_{i=1}^{N_p}\delta_i^p a(\tau_i^p)(s-S(0)) + 1 =& 1 \\
\Sigma_{i=1}^{N_p}\delta_i^p a(\tau_i^p)(s-S(0)) =& 0 \\
\Sigma_{i=1}^{N_p}\delta_i^p a(\tau_i^p) =& 0
\end{aligned}$$

Condition 3. Definition of swap rate in terms of bond prices.

$$\begin{aligned}
S(T) =& \frac{\Sigma_{i=1}^{N_r} L_i(T,\tau_{i-1}^r,\tau_i^r)\delta^r P(T,\tau_i^r)}{\text{Ann}(T)} \\
=& \frac{P(T,\tau_{N_p}^p)-1}{\text{Ann}(T)}
\end{aligned}$$

This implies

$$
\begin{aligned}
\mathbb{E}^{\text{Ann}}[S(T)|S(T)=s] =& \mathbb{E}^{\text{Ann}}[\frac{P(T,\tau^p_{N_p})-1}{\text{Ann}(T)}|S(T)=s] \\
s =& M(s,\tau^p_{N_p}) - M(s,T) \\
=& (a(\tau^p_{N_p}) - a(T_p))(s - S(0)) + \frac{P(0,\tau^p_{N_p})-1}{\text{Ann}(0)} \\
=& (a(\tau^p_{N_p}) - a(T_p))(s - S(0)) + S(0) \\
s - S(0) =& (a(\tau^p_{N_p}) - a(T_p))(s - S(0)) \\
a(\tau^p_{N_p}) - a(T_p) =& 1
\end{aligned}
$$

From the implications of conditions 2 and 3, we can determine the coeffecients u and v in the linear map $a(t) = ut + v$, and that would fully specify our linear annuity mapping. We can then compute price of CMS caplet using a terminal swap rate model, without needing a full term-structure model such as Hull-White.

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