Holographic Entanglement Entropy

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by

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Approval Sheet

This report entitled 'Holographic Entanglement Entropy' by Sarthak Bagaria is approved for the degree of Bachelor of Technology.

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Contents

Abstract

In this report, we review a derivation of the entropy formula for holographic gravitational theories as was found in [1]. We also study its extention to prove the Ryu-Takayanagi [2] conjecture which states that the entropy of a region of a field theory is proportional to the area of the minimal surface in the dual gravity theory with the surface ending on the boundary of the field theory region.

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Chapter 1

Introduction

In this report we review a derivation of the gravitational entropy in a Euclidean theory without the U(1) symmetry. For the case with U(1) symmetry, the entropy was calculated by Hawking and Gibbons [3]. Our approach is based on the replica trick method which requires calculation of partition function of the theory on an n-replica manifold. The construction of the replica manifold is described in the report. Crucial in this construction is the role of holography. We take our theory to be holographic and believe that setting the boundary conditions describes the theory completely. Even though the method gives valid boundary conditions only for integer n, we nevertheless calculate the values for integers and analytically continue them to non-integer values, which is the essence of the replica trick.

The calculations for entropy are based on the presence of a co-dimension 2 surface which is a fixed point of the Z_n symmetry of the replica. The Euclidean time circle shrinks to zero on this surface, and we get a singularity. The singularity differs from conical singularity in that it may not have the O(2) symmetry of the cone. The curvature scalar obtains a delta peak at this singularity which upon integration in the gravity action contributes the area term to the entropy. The area is of the singular surface which may be seen to be a minimal surface when the system is restricted to obey Einstein equations close to the surface in leading order of n-1.

At last we show how the analysis of gravitational entropy can be extended to calculate entanglement entropy in dual field theories. The result matches with the Ryu-Takayanagi conjecture.

Chapter 2

Partition Function

In this chapter we relate the partition function of our theory to a path integral, and write the expressions for density matrix and the entropy in terms of the partition function. We also introduce the replica trick for computation of entanglement entropy.

2.1 Path Integral

In the path integral approach, we have for a field evolving from ϕ_1 at time t_1 to ϕ_2 at t_2 [3],

$$\langle \phi_2, t_2 | \phi_1, t_1 \rangle = \int d[\phi] \exp(\iota I[\phi]) \tag{2.1}$$

where the integral is over all fields which satisfy the given field values at times t_1 and t_2. We also have

$$\langle \phi_2, t_2 | \phi_1, t_1 \rangle = \langle \phi_2 | \exp[-\iota \int_{t_1}^{t_2} H dt] | \phi_1 \rangle. \tag{2.2}$$

Taking the integral on the Euclidean section (with $dt = -\iota d\tau$), and setting $\phi_1 = \phi_2$ we have,

$$Z = \mathrm{Tr} \exp(-\int_{\tau_1}^{\tau_2} H d\tau) = \int d[\phi] \exp(\iota I[\phi]) \tag{2.3}$$

where the integral is over the fields periodic with period $\tau_2 - \tau_1$, which from now on we set to be equal to 2π. The left side of the last equality can be interpreted as the partition function of a thermal state.

2.2 Entropy

From the theory of statistical mechanics, we take the density matrix for the Euclidean theory to be

$$\hat{\rho} = \frac{1}{Z}\exp(-\int_{\tau_1}^{\tau_2} H d\tau) \equiv \frac{1}{Z}\rho \tag{2.4}$$

$$\langle\phi_2|\rho|\phi_1\rangle = \frac{1}{Z}\int_{\phi_1}^{\phi_2} d[\phi]\exp(\iota I[\phi]) \tag{2.5}$$

where ρ is the un-normalized density matrix.

The entanglement entropy or the Von-Neumann entropy for the system in defined as

$$S = -\mathrm{Tr}[\hat{\rho}\log(\hat{\rho})] \tag{2.6}$$

which may be written as

$$S = -\lim_{n\to 1}\frac{\partial}{\partial n}\mathrm{Tr}\left[(\frac{\rho}{Z})^n\right] = -\lim_{n\to 1} n\partial_n[\log(Z(n)) - n\log(Z(1))] \tag{2.7}$$

where $Z(n) = \mathrm{Tr}[\rho^n]$ and $Z(1) = Z$.

2.3 Replica Trick

In holographic theories, setting the boundary conditions describes the entire system. That is, the path integral in the bulk is equal to the partition function on the boundary. This fact is crucial in construction of our replica manifold, as the replica construction of the boundary theory can induce the proper interior geometry in the bulk.

For integer n, the formula for $Z(n)$ can be found from the path integral in a manifold $\mathcal{M}_n$ which consists of n replicas of the original manifold. Notice that

$$Z(n) = \mathrm{Tr}[\rho^n] = \langle\phi_1|\rho|\phi_2\rangle\langle\phi_2|\rho|\phi_3\rangle\ldots\langle\phi_n|\rho|\phi_1\rangle \tag{2.8}$$

where we have used the completeness relation. Thus the geometry for $Z(n)$ can be constructed from n replicas of the original geometry in the following manner. Introduce a cut along the $t = 0$ hyper-surface of the original manifold $\mathcal{M}$. Construct n replicas of the manifold. Join the t_+ boundary of the cut of i'th replica to t_- boundary of the cut of (i+1)'th replica for $1 \leq i \leq n-1$. Join the t_+ boundary of the cut of n'th replica to t_- boundary of the cut of 1st replica. The obtained manifold is $\mathcal{M}_n$.

The length of time circle in the new manifold is $2\pi n$ as compared to 2π in the original manifold. But the couplings are periodic with period 2π as they would be same for each of the replica. That is, we have Z_n symmetry on the boundary. We assume here that the Z_n symmetry extends to the bulk solution.

When only discreet time symmetry is present instead of continuous symmetry, the above construction would not hold any meaning for non-integer n, but we nevertheless analytically continue our expression to non-integer values to obtain the entropy.

Chapter 3

Gravitational Entropy

For $n > 1$, we have a special co-dimension 2 surface Σ which is a fixed point of the Z_n symmetry and where the time circle shrinks to zero. This surface induces a singularity in the metric. If the theory had continuous time symmetry, the singularity would be conical. But for the general case, we have the discrete symmetry $\tau \to \tau + 2\pi k$ for integer k, and we call singularities of this kind *squashed cones* [4]. In this chapter we study the properties of this surface singularity and how it contributes to entanglement entropy.

Note that if there were no such special surface, where the circle shrinks to 0, then $\log(Z(n))$ would simply be equal to $n \log(Z(1))$ owing to the symmetry of the system, and hence giving a zero entropy.

3.1 Metric

For a static space-time, the metric may be written as

$$ds^2 = B(x)dt^2 + h_{ab}(x)dx^a dx^b \qquad\qquad a, b = 1, \ldots, d-1. \tag{3.1}$$

We consider only the metrics with Euclidean signature and hence take $B(x) > 0$. The co-dimension 2 surface Σ has only one non-vanishing extrinsic curvature for static spacetimes, since the extrinsic curvature for a normal vector directed along the Killing vector ∂_t is zero.

Let $\mathcal{H}$ be a constant time hyper-surface. Consider in $\mathcal{H}$ the normal Riemann coordinates $r, y^i (i = 1, \ldots, d-2)$ with origin on Σ. We then have

$$h_{ab}(x)dx^a dx^b = d\varrho^2 + (\gamma_{ij}(y) + 2\varrho k_{ij}(y) + O(\varrho^2))dy^i dy^j. \tag{3.2}$$

Coordinate ϱ is the geodesic distance from a point on the hypersurface $\mathcal{H}$ to Σ and $\gamma_{ij}(y)dy^i dy^j$ is a metric on Σ. Then $k_{ij}(y)$ is the extrinsic curvature tensor of Σ for the unit normal vector $n_a = \delta_a^r$. Introducing the coordinate $\zeta = \sqrt{B}t$,

$$d\zeta = \sqrt{B}dt + \zeta w_a dx^a, \tag{3.3}$$

where $w_a = \frac{1}{2}\partial_a B/B$ are the acceleration vector of the coordinate frame. Then the metric up to second order in ϱ and ζ takes the form

$$ds^2 \simeq d\zeta^2 + d\varrho^2 + (\gamma_{ij}(y) + 2\varrho k_{ij}(y))dy^i dy^j - 2\zeta w_\varrho(y) d\zeta d\varrho - 2\zeta d\zeta w_i(y) dy^i. \tag{3.4}$$

We make the coordinate transformation

$$v^i = y^i - \frac{1}{2}\zeta^2 w^i(y), \qquad \bar{\varrho} = \varrho - \frac{1}{2}\zeta^2 w_\varrho(y) \tag{3.5}$$

to get

$$ds^2 \simeq dx_1^2 + dx_2^2 + (\gamma_{ij}(v) + 2x_2 k_{ij}(v))dv^i dv^j, \tag{3.6}$$

where $x_1 = \zeta$ and $x_2 = \bar{\varrho}$ and terms second order in ζ and $\bar{\varrho}$ are omitted. We again make the transformation

$$x_1 = r\sin\tau, \qquad x_2 = r\cos\tau \tag{3.7}$$

to get the metric in the form

$$ds^2 \simeq r^2 d\tau^2 + dr^2 + (\gamma_{ij}(v) + 2r\cos(\tau)k_{ij}(v))dv^i dv^j. \tag{3.8}$$

In static space-time the surface Σ has a single non-vanishing extrinsic curvature. Generalization to surfaces with two non-trivial extrinsic curvatures gives

$$ds^2 \simeq dx_1^2 + dx_2^2 + (\gamma_{ij}(v) + 2x_p k_{ij}^{(p)}(v))dv^i dv^j, \tag{3.9}$$

$$ds^2 \simeq r^2 d\tau^2 + dr^2 + (\gamma_{ij}(v) + 2r\cos(\tau)k_{ij}^{(1)}(v) + 2r\sin(\tau)k_{ij}^{(2)}(v))dv^i dv^j. \tag{3.10}$$

3.2 Linearized Equations of Motion

We obtain conditions the singular surface must satisfy so that for small $\epsilon \equiv n-1$ the system obeys the linearized field equations near $r=0$ i.e., the system obeys Einstein equations to leading order in $n-1$ [1].

The modification in metric up to linear order in ϵ upon application of replica trick and analytic continuation is

$$ds^2 = e^{2\rho}(dr^2 + r^2 d\tau^2) + g_{ij} dv^i dv^j + \delta g \tag{3.11}$$

$$g_{ij} = \gamma_{ij}(v) + 2x_1 k_{ij}^{(1)} + 2x_2 k_{ij}^{(2)} \tag{3.12}$$

where the factor $e^{2\rho}$, $\rho \sim \epsilon \log r$ to first order has been introduced to make the metric smooth near $r = 0$. We now work in the complex coordinates $z = x_1 + \iota x_2$. As gauge conditions we set $\delta g_{zz} = \delta g_{\bar{z}\bar{z}} = 0$. We also set $\delta g_{z\bar{z}} = 0$ as this variation is already already included in ρ. We require the perturbation to be periodic with $\delta g_{ab}(\tau) = \delta g_{ab}(\tau + 2\pi)$.

We expect the v_i derivatives to be regular and only x_i derivates to contribute to divergences in the curvatures. The system must satisfy the linearized equations $\delta G_{zz} = \delta T_{zz}$. We have

$$\delta R_{zz} = \frac{-\epsilon}{z} 2k_z + \frac{1}{2}(2\delta g^p_{z;zp} - \delta g_{;zz} - \nabla^2 \delta g_{zz}) + (\text{regular as } r \to 0) \quad (3.13)$$

$$= \frac{-\epsilon}{z} 2k_z - \frac{1}{2}\partial_z^2 \delta h + \ldots \quad (3.14)$$

where $\delta h = g^{ij}\delta g_{ij}$ and $k_z = k^1 - \iota k^2$. To avoid singularity in the stress tensor the two potential divergent terms must be cancel

$$\frac{1}{2}\partial_z^2 \delta h = \frac{-\epsilon}{z} 2k_z \quad (3.15)$$

$$\frac{1}{2}\partial_{\bar{z}}^2 \delta h = \frac{-\epsilon}{\bar{z}} 2k_{\bar{z}}. \quad (3.16)$$

On imposing the condition that δh is periodic in time, we get $k_z = k_{\bar{z}} = 0$ which implies that the extrinsic curvature of the singular surface is 0 and hence the surface is a minimal area surface. To check this observe that δh is periodic in τ and so are its derivates, and that the time integral of time derivative of a periodic function is 0. Hence integral of,

$$\partial_t[(r\partial_r - 1)\partial_z \delta h] \propto (z\partial_z - \bar{z}\partial_{\bar{z}})(z\partial_z + \bar{z}\partial_{\bar{z}} - 1)\partial_z \delta h \propto \epsilon k_z \quad (3.17)$$

is zero. To get the above proportionality, observe that k_z and $k_{\bar{z}}$ are independent of z and $\bar{z}$ and hence we have the following relations

$$\partial_z^3 \delta h = -\partial_z(\frac{4\epsilon k_z}{z}) = \frac{4\epsilon k_z}{z^2} \quad (3.18)$$

$$\partial_{\bar{z}}\partial_z^2 \delta h = -\partial_{\bar{z}}(\frac{4\epsilon k_z}{z}) = 0 \quad (3.19)$$

$$\partial_z\partial_{\bar{z}}^2 \delta h = -\partial_z(\frac{4\epsilon k_{\bar{z}}}{\bar{z}}) = 0 \quad (3.20)$$

$$\partial_{\bar{z}}^3 \delta h = -\partial_{\bar{z}}(\frac{4\epsilon k_{\bar{z}}}{\bar{z}}) = \frac{4\epsilon k_{\bar{z}}}{\bar{z}^2}. \quad (3.21)$$

Now, since k_z is independent of τ and its τ integral is 0, we have $k_z = 0$. Similarly $k_{\bar{z}} = 0$.

3.3 Area Term in Entropy

The gravity action for the n-replica in a d+1 dimension space-time is given by

$$I_{\mathrm{gr},\mathcal{M}_n} = -\frac{1}{16\pi G_N^{(d+1)}}\int_{\mathcal{M}_n} d^{d+1}x\sqrt{g}(R+2\Lambda) - \frac{1}{8\pi G_N^{(d+1)}}\int_{\partial\mathcal{M}_n} d^d x\sqrt{h}K \quad (3.22)$$

where Λ is the negative cosmological constant and the last term is the Gibbons-Hawking boundary term. Under saddle point analysis, the path integral can be taken to be the extremal action given by the fields satisfying the equations of motion.

We know from the previous section that the extrinsic curvature vanishes for the singular surface. Hence, up-to second order in r, the metric is time independent and the singularity is approximately conical. The presence of the singular surface gives rise to a peak in the curvature scalar at the singular surface which gives a contribution to the action [4]

$$\int_{\mathcal{M}_n}\sqrt{g}d^4xR \to n\int_{\mathcal{M}}\sqrt{g}d^4xR + 4\pi(1-n)A(\Sigma) + \ldots \quad (3.23)$$

where $A(\Sigma)$ is the area of the singular surface Σ, and the regularization dependent $O((n-1)^2)$ terms have been omitted.

Therefore,

$$\log(Z(n)) - n\log(Z(1)) = \frac{(1-n)A(\Sigma)}{4G_N^{(4)}} \quad (3.24)$$

$$S = \frac{A(\Sigma)}{4G_N^{(4)}} \quad (3.25)$$

From the above analysis the entropy may be seen to be based on the local property of the fixed point surface with vanishing curvature.

We now provide a proof of relation (3.23) for conical singularities [5]. The metric on a space $\mathcal{M}_\alpha$ with topology of cone C_α is given by

$$ds^2 = e^{\sigma}(d\rho^2 + \rho^2 d\phi^2) \equiv e^{\sigma}ds_C^2 \quad (3.26)$$

where ds_C^2 is the line elements on C_α, which upon hyperbolic regularization by a parameter a gives

$$ds_H^2 = u d\rho^2 + \rho^2 d\phi^2 = \frac{\rho^2 + a^2\alpha^2}{\rho^2 + a^2}d\rho^2 + \rho^2 d\phi^2. \quad (3.27)$$

Representing the scalar curvature as

$$R = e^{-\sigma} R_H - e^{-\sigma} \Box_H \sigma \tag{3.28}$$

where R_H and $\Box_H$ are the curvature and Laplace operator defined with respect to hyperbolic metric, and taking the volume element $d\mu = e^{\sigma}\sqrt{u}\rho d\rho d\phi$, we evaluate the scalar curvature

$$\int_{\bar{\mathcal{M}}_\alpha} R = 2\pi\alpha \int_0^\infty d\rho u'_\rho u^{-\frac{3}{2}} - \int_0^\infty \int_0^{2\pi\alpha} \sqrt{u}\rho d\rho d\phi \Box_H \sigma \tag{3.29}$$

$$= 4\pi(1-\alpha) - \int_0^\infty \int_0^{2\pi\alpha} \sqrt{u}\rho d\rho d\phi \Box_H \sigma \tag{3.30}$$

where $\bar{\mathcal{M}}_\alpha$ is the regularized manifold. The first term is related to the Euler number of the surface Σ and is a topological characteristic independent of the parametrization. The second term is parametrization dependent but in the limit $a \to 0$, where regularization is taken off, we have

$$\lim_{\bar{\mathcal{M}}_\alpha \to \mathcal{M}_\alpha} \int_{\bar{\mathcal{M}}_\alpha} R = 4\pi(1-\alpha) + \int_{\mathcal{M}_\alpha/\Sigma} R \tag{3.31}$$

where σ is the singular region. In higher dimensions we have

$$\int_{\mathcal{M}_\alpha} R = 4\pi(1-\alpha)A(\Sigma) + \int_{\mathcal{M}_\alpha/\Sigma} R. \tag{3.32}$$

where $A(\Sigma)$ is the area of Σ. Since only the singular surface gives rise to the first term on the right side of equation, we see can consider a local representation of curvature with a peak on singular surface.

Chapter 4

Entanglement Entropy in Dual Field Theories

For calculation of entanglement entropy, we are interested in entanglement among fields in two spatial regions (accessible A and inaccessible B) separated by a co-dimension 2 surface Σ at a specific time. The $t = 0$ cut in this case is along the co-dimension 1 surface which is accessible to the observer and has Σ as its boundary. The cut on the inaccessible region disappears while tracing over that region to obtain the reduced density matrix.

We consider field theories with gravity duals. The field theory lives at the boundary of a one dimension higher gravity theory. The cut A in the field theory region induces a cut $\tilde{A}$ in the bulk with a boundary $\tilde{\Sigma}$ which is homologous to A, $A \cup \tilde{\Sigma} = \partial \tilde{A}$ [6]. $\tilde{\Sigma}$ would be a singular surface in the bulk of the replica manifold. From the analysis of previous section, the extrinsic curvatures of the singular surface must vanish for the system to obey linearized equations of motion. We then expect the boundary $\tilde{\Sigma}$ of the cut to be a minimal surface satisfying the criteria that it be homologous to the field theory region with $\partial\tilde{\Sigma} = \Sigma$.

The entropy is then given by the area of the above minimal surface as in the previous section, and we thus obtain the entropy formula conjectured by Ryu and Takayanagi [2].

Chapter 5

Conclusion

In this report we used the replica trick, which is a standard technique for computing entanglement entropy in field theories [7], to calculate entropy in holographic gravitational theories. The entanglement entropy in gravitational theory matches with the Bekenstein-Hawking entropy for black holes and suggests that black hole entropy may be an entanglement entropy.

We also extended the analysis of holographic gravitational entropy to calculate entanglement entropy of dual field theories on the boundary of gravity theory. The result matches with the Ryu-Takayanagi conjecture and connects the entropy calculations in quantum field theory to calculations in classical geometry.

Our analysis in this report was based on Euclideanization of static space-time. The surfaces considered were embedded in constant time hypersurfaces. Ryu-Takayanagi conjecture is however predicted to be true for surfaces in full Lorentzian space-time. A generalized derivation of entropy for time dependent cases may require an approach other than Euclideanization.

One point worthwhile to think about would be whether the the singular surface is always homologous to the boundary cut in holographic gravity theories. For example, if the boundary is closed the singular surface may be the horizon of a black hole contained in the bulk, and in this case the black hole boundary seems to be homologous to the boundary cut. If the property would hold in general, Ryu-Takayanagi conjecture would be a simple consequence of this property. If true, it may also give nice geometric interpretations of entropy in holographic theories.

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