Planar Limit Analysis of Matrix Models

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by

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Approval Sheet

This report entitled 'Planar Limit Analysis of Matrix Models' by Sarthak Bagaria is approved for the degree of Bachelor of Technology.

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Contents

Abstract

In this report, we review some basic techniques used in the theory of random matrices and apply them to solve 2-matrix models. Specifically, we study, for large matrix dimension and weak coupling, the Lens Space model which corresponds to the Chern Simons theory on $S^3/\mathbb{Z}_2$. We also extend some of the results to the ABJM matrix model.

Chapter 1

Introduction

A random matrix is a matrix with random entries. The distributions they follow are generally inferred from the symmetries involved in the problems. Random matrices have long been used in physics to study, among others, the energy level spacings in nuclei, chaotic quantum systems and 2-dimensional quantum gravity [1]. Certain matrix models have also been shown to represent non-perturbative formulations of String Theory with low space-time dimensions [2, 3]. Many techniques have been employed to solve matrix models, some of them being method of orthogonal polynomials, saddle-point analysis, and Feynman perturbative methods [4].

In Chapter 2, we review some of the basic concepts of the random matrix theory. We study the reduction of the matrix measure to the eigenvalue space for Hermitian matrices with action only dependent on the eigenvalues. We then use saddle-point analysis to get the *planar* eigenvalue distribution. The term planar has been used because saddle-point analysis gives, in the limit of large matrix dimension, the *genus zero* contribution to the expression which can be obtained by summing the contributions from the planar diagrams in the Feynman's diagrammatic method. We also give an expression for the planar free energy in terms of the eigenvalue distribution, which may further be reduced to certain contour integral of the resolvent [5].

Two-matrix models involve two random matrices with effective potentials that couple the two matrices. Chern-Simons theory on Lens Space L(2,1) ($S^3/\mathbb{Z}_2$) has been shown to reduce to a Hermitian 2-matrix model with unitary Haar measure [6]. We study this matrix model in Chapter 3. We show how the one-matrix model saddle-point analysis can be extended two-matrix problem and obtain a closed form solution for the planar resolvent of the lens space model. We then use perturbative calculations to explicitly compute the 't Hooft parameters in terms of the branch cut endpoints in the weak coupling regime. It has also been shown that the path integral in the calculations of expectation values of supersymmetric

Wilson loops in Chern-Simons theories with matter reduce to matrix model [7]. Since this matrix model for the ABJM theory is related to the lens space model by an analytic continuation of one of the 't Hooft parameters, we easily extend the results from lens space model to calculate the vacuum expectation value of the 1/2 BPS Wilson Loop.

Chapter 2

Random Matrices

In this chapter, we study the formal aspects of Hermitian one-matrix models with a unitary Haar measure. We derive the planar eigenvalue distribution and free energy using the saddle-point analysis and resolvent formalism.

2.1 Basics

Consider the following action formed from Hermitian $N \times N$ matrix M [8]:

$$\frac{1}{g_s}W(M) = \frac{1}{2g_s}Tr(M^2) + \frac{1}{g_s}\sum_{p\geq 3}\frac{g_p}{p}Tr(M^p). \tag{2.1}$$

The action has a gauge symmetry

$$M \to UMU^\dagger \tag{2.2}$$

and the partition function for the theory is given by

$$Z = \frac{1}{vol(U(N))}\int dM\, e^{-\frac{1}{g_s}W(M)} \tag{2.3}$$

The measure in the integral is the unitary Haar measure ($dM = d(UM)$ for every unitary matrix U)

$$dM = 2^{\frac{N(N-1)}{2}}\prod_{i=1}^{N} dM_{ii} \prod_{1\leq i<j\leq N} d\mathrm{Re}(M_{ij})\, d\mathrm{Im}(M_{ij}) \tag{2.4}$$

and $vol(U(N))$ is the volume factor of the gauge group.

2.2 Eigenvalue Space

The Hermitian matrix M can be reduced to a diagonal form [3]

$$M = U\Lambda U^\dagger \tag{2.5}$$

where U is a unitary matrix and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$. Now we have

$$\begin{aligned} dM &= dU\,\Lambda\,U^\dagger \;+\; U\,d\Lambda\,U^\dagger \;+\; U\,\Lambda\,dU^\dagger & (2.6)\\ \Rightarrow\; U^\dagger\,dM\,U &= d\Lambda \;+\; [U^\dagger\,dU,\,\Lambda] & (2.7)\end{aligned}$$

$d\alpha = U^\dagger dU$ is the infinitesimal element in the tangent space to the unitary group, the measures $[dM]$ and $[dM'] = [U^\dagger\,dM\,U]$ are the same, and

$$dM'_{ij} = d\lambda_i \delta_{ij} + d\alpha_{ij}(\lambda_i - \lambda_j) \tag{2.8}$$

upon change of coordinates from M'_{ij} to $(\lambda_i, \alpha_{ij(i\neq j)})$. Therefore,

$$[dM] \;=\; \prod_{i\neq j} |(\lambda_i - \lambda_j)|\, d\alpha_{ij} \prod_{i=1}^{N} d\lambda_i. \tag{2.9}$$

The factor $\Delta(\lambda)^2 = \prod_{i\neq j} |(\lambda_i - \lambda_j)|$ is known as the *Vandermonde determinant.* Since, our original action only included traces of powers of the matrix M, the integrand can be represented only in the terms of the eigenvalues, and the integral over $d\alpha$ is just a numerical factor.

2.3 Saddle-Point Analysis

The partition function can be now be written as [5]

$$Z = \frac{1}{N!}\int \prod_{i=1}^{N} \frac{d\lambda_i}{2\pi}\, e^{g_s^{-2} S_{\mathrm{eff}}(\lambda)} \tag{2.10}$$

where the effective action S_{eff} is given by

$$S_{\mathrm{eff}}(\lambda) = -\frac{t}{N}\sum_{i=1}^{N} V(\lambda_i) + \frac{2t^2}{N^2}\sum_{i<j} \log|\lambda_i - \lambda_j| \tag{2.11}$$

with the 't Hooft parameter $t = g_s N$.

As the summation over N is roughly of order $O(N)$, the effective action can be seen to be $O(1)$ in the large N limit. In the limit $g_s \to 0$, the contribution to the partition function will be dominated by the *saddle-point* configuration that extremizes the action. This is analogous to attaining lowest energy configuration in the limit of zero temperature. We obtain the saddle-point equations by varying the effective action with respect to the eigenvalues

$$\frac{1}{2t}V'(\lambda_i) = \frac{1}{N}\sum_{j\neq i}\frac{1}{\lambda_i - \lambda_j}, \qquad i = 1, \ldots, N. \tag{2.12}$$

We may interpret the λ_i as the coordinates of charged particles moving in an effective potential

$$V_{\text{eff}} = V(\lambda_i) - \frac{2t}{N}\sum_{j\neq i} log|\lambda_i - \lambda_j| \tag{2.13}$$

where the logarithmic term acts as coulomb repulsion. Near $t = 0$ the potential term dominates over Coulomb interaction, and the eigenvalues lie close to a critical point with $V'(x^*) = 0$. As t grows, the Coulomb interaction starts spreading out the eigenvalues. The distribution of eigenvalues may be written as

$$\rho(\lambda) = \frac{1}{N}\sum_{i=1}^{N}\delta(\lambda - \lambda_i) \tag{2.14}$$

where λ_i are taken from the solutions of the saddle-point equations. In the large N limit, we expect the distribution to form a continuum around the critical point x^* of the potential. If the support is taken to be such an interval, it is known as *one-cut solution.*

Using the continuum approximation

$$\frac{1}{N}\sum_{i=1}^{N} f(\lambda_i) \to \int_C f(\lambda)\rho_0(\lambda)d\lambda, \tag{2.15}$$

we may write the saddle-point equation as

$$\frac{1}{2t}V'(\lambda) = P\int_C \frac{\rho_0(\lambda')d\lambda'}{\lambda - \lambda'} \tag{2.16}$$

where P denotes the principal value of the integral and ρ_0 is the distribution in the large N limit. One standard way to solve the integral equation for ρ_0 is to introduce an auxiliary function called the *resolvent*, which is defined as

$$\omega(p) = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p - \lambda_i} \tag{2.17}$$

and labelled $\omega_0(p)$ in the limit of large N. The genus zero resolvent has a discontinuity along interval C (the support of the eigenvalue distribution). The discontinuity can be calculated by contour deformations.

$$\begin{aligned}\omega_o(p+\iota\epsilon) &= \int_{\mathbb{R}} d\lambda \frac{\rho_0(\lambda)}{p+\iota\epsilon-\lambda} = \int_{\mathbb{R}-\iota\epsilon} d\lambda \frac{\rho_0(\lambda)}{p-\lambda} &(2.18)\\ &= P\int d\lambda \frac{\rho_0(\lambda)}{p-\lambda} + \int_{C_\epsilon} d\lambda \frac{\rho_0(\lambda)}{p-\lambda}, &(2.19)\end{aligned}$$

where C_ϵ is a counterclockwise oriented contour around $\lambda = p$ in the lower half plane. The last integral can be evaluated as a residue, and we have

$$\omega_0(p+\iota\epsilon) = P\int d\lambda \frac{\rho_0(\lambda)}{p-\lambda} - \pi\iota\rho_0(p). \tag{2.20}$$

Similarly,

$$\omega_0(p-\iota\epsilon) = \int_{\mathbb{R}+\iota\epsilon} d\lambda \frac{\rho_0(\lambda)}{p-\lambda} = P\int d\lambda \frac{\rho_0(\lambda)}{p-\lambda} + \pi\iota\rho_0(p). \tag{2.21}$$

Hence, we find

$$\begin{aligned}\omega_0(\lambda+\iota\epsilon) - \omega_0(\lambda-\iota\epsilon) &= -2\pi\iota\rho_0(\lambda) &(2.22)\\ \omega_0(p+\iota\epsilon) + \omega_0(p-\iota\epsilon) &= P\int d\lambda \frac{\rho_0(\lambda)}{p-\lambda} = \frac{i}{t}V'(p) &(2.23)\end{aligned}$$

where the last equality follows from (2.16). The problem of finding the eigenvalue distribution has reduced to the Riemann-Hilbert problem of computing the resolvent [8].

2.4 Free Energy

The free energy of the matrix model is given by

$$F = \log Z. \tag{2.24}$$

As can be seen from (2.10), in the limit $g_s \to 0$, the free energy scales as

$$F(g_s, t) \approx g_s^{-2} F_0(t) \tag{2.25}$$

where $F_0(t)$ is the genus zero, or planar, free energy given by

$$F_0(t) = S_{\text{eff}}(\rho_0) \tag{2.26}$$

and

$$S_{\text{eff}}(\rho_0) = -t\int_C d\lambda \rho_0(\lambda) V(\lambda) + t^2 \int_{C\times C} d\lambda d\lambda' \rho_0(\lambda)\rho_0(\lambda') \log|\lambda - \lambda'| \tag{2.27}$$

is the continuum limit of (2.11).

Chapter 3

Two-Matrix Models

In this chapter, we extend the saddle-point and resolvent formalism from the one-matrix model solution in previous chapter to two-matrix models, specifically lens space model in this report. We illustrate a perturbative calculation to obtain the branch cuts in terms of the 't Hooft parameters. We also use the 't Hooft parameter dictionary between the lens space matrix model and ABJM matrix model to calculate the vacuum expectation values of Wilson Loops.

3.1 Solving the Lens Space Model

The partition function for the Chern Simons theory on the $S^3/\mathbb{Z}_2$ is given by [9]

$$Z \sim \int \prod_i du_i \prod_\alpha d\mu_\alpha \, \Delta^2(u,\mu) \, \exp\big(- \frac{1}{g_s} V(u,\mu)\big) \qquad i \in (1, N_1), \alpha \in (1, N_2) \tag{3.1}$$

where the measure

$$\Delta(u,\mu) = \prod_{i<j} 2\sinh\left(\frac{u_i - u_j}{2}\right) \prod_{\alpha<\beta} 2\sinh\left(\frac{\mu_\alpha - \mu_\beta}{2}\right) \prod_{i,\alpha} 2\cosh\left(\frac{u_i - \mu_\alpha}{2}\right) \tag{3.2}$$

and the potential is

$$V(u,\mu) = \Big(2\sum_i u_i^2 \, + \, 2\sum_\alpha \mu_\alpha^2\Big)/2 \tag{3.3}$$

In the limit of large N and for weak coupling, we obtain from saddle-point analysis, by varying the action with respect to u_i and μ_α

$$2u_i = g_s \sum_{j\neq i} \coth\left(\frac{u_i - u_j}{2}\right) + g_s \sum_\alpha \tanh\left(\frac{u_i - \mu_\alpha}{2}\right) \tag{3.4}$$

$$2\mu_\alpha = g_s \sum_{\beta\neq\alpha} \coth\left(\frac{\mu_\alpha - \mu_\beta}{2}\right) + g_s \sum_i \tanh\left(\frac{\mu_\alpha - u_i}{2}\right) \tag{3.5}$$

Observing the above the saddle-point equations, and proceeding in a similar manner as for one-matrix model, we define the resolvent as

$$\omega(z) = g_s \sum_i \coth\Big(\frac{z-u_i}{2}\Big) + g_s \sum_\alpha \tanh\Big(\frac{z-\mu_\alpha}{2}\Big). \tag{3.6}$$

We may rewrite this as

$$\omega(z) = \omega^{(1)}(z) + \omega^{(2)}(z-\iota\pi) \tag{3.7}$$

with

$$\omega^{(1)}(z) = g_s \sum_i \coth\Big(\frac{z-u_i}{2}\Big) \tag{3.8}$$

$$\omega^{(2)}(z) = g_s \sum_\alpha \coth\Big(\frac{z-\mu_\alpha}{2}\Big). \tag{3.9}$$

On multiplying (3.4) by $\coth((z-u_i)/2)$ and summing over i, as well as multiplying (3.5) by $\tanh((z-\mu_\alpha)/2)$ and summing over α, we obtain for large N limit

$$\Big(\frac{\omega_0(z)}{2}\Big)^2 - 2z\frac{\omega_0^{(1)}(z)}{2} - 2(z-\iota\pi)\frac{\omega_0^{(2)}(z-\iota\pi)}{2} = f(z) \tag{3.10}$$

where

$$f(z) = g_s \sum_i (u_i - z)\coth\Big(\frac{z-u_i}{2}\Big) + g_s \sum_\alpha (\mu_\alpha - (z-\iota\pi))\tanh\Big(\frac{z-\mu_\alpha}{2}\Big) + \frac{1}{4}S^2 \tag{3.11}$$

is a regular function. (3.10) may be written in two ways,

$$\Big(\frac{\omega_0(z)}{4}\Big)^2 - (z-\iota\pi)\frac{\omega_0(z)}{4} - \iota\pi\frac{\omega_0^{(1)}(z)}{4} = \frac{f(z)}{4} \tag{3.12}$$

$$\Big(\frac{\omega_0(z+\iota\pi)}{4}\Big)^2 - (z+\iota\pi)\frac{\omega_0(z+\iota\pi)}{4} + \iota\pi\frac{\omega_0^{(2)}(z)}{4} = \frac{f(z+\iota\pi)}{4} \tag{3.13}$$

Assuming that the eigenvalues only spread along the real line, it follows that if $\omega_1(z)$ jumps at a point z then $\omega_2(z-\iota\pi)$ does and vice versa. The relative shift in the arguments of the two resolvents gives a separation of $\iota\pi$ between the two cuts. On one cut the resolvent jumps due to $\omega_1(z)$ alone and due to $\omega_2(z)$ alone on the second cut. From this we can deduce, in a manner similar to one we used to derive (2.23)

$$z = \frac{1}{4}(\omega_0(z+\iota\epsilon) + \omega_0(z-\iota\epsilon)) \quad \text{(u cut)} \tag{3.14}$$

$$z = \frac{1}{4}(\omega_0(z+\iota\pi+\iota\epsilon) + \omega_0(z+\iota\pi-\iota\epsilon)) \quad (\mu \text{ cut}) \tag{3.15}$$

which implies that the resolvent has square root branch cuts (observing (2.23) and that $x = \frac{1}{2}x^{2\prime})$.

Consider the function

$$g(Z) \equiv e^{\omega_0/2} + Z^2 e^{-\omega_0/2} \tag{3.16}$$

where $Z = e^z$. From (3.6), we see that

$$\lim_{z\to\infty} \omega_0 = \quad g_s N_1 + g_s N_2 = t_1 + t_2 \quad = t \tag{3.17}$$
$$\lim_{z\to-\infty} \omega_0 = \quad -g_s N_1 - g_s N_2 = -t_1 - t_2 \quad = -t. \tag{3.18}$$

Therefore, the function $g(Z)$ has the limiting behaviour

$$\lim_{Z\to\infty} g(Z) \quad = \quad Z^2 e^{-t/2} \tag{3.19}$$
$$\lim_{Z\to 0} g(Z) \quad = \quad e^{-t/2}. \tag{3.20}$$

The unique solution satisfying these conditions is

$$g(Z) = e^{-t/2}(Z^2 - \zeta Z + 1) \tag{3.21}$$

where ζ is to be determined. Inverting $g(Z)$ we find

$$\frac{\omega_0(Z)}{2} = \log\Big(\frac{1}{2}\big[g(Z) - \sqrt{g^2(Z) - 4Z^2}\big]\Big) \tag{3.22}$$

We can easily see that e^{ω_0} has a square root branch cut involving the function

$$\sigma(Z) = g^2(Z) - 4Z^2 = e^{-t}(Z-a)(Z-1/a)(Z+b)(Z+1/b). \tag{3.23}$$

$a^{\pm 1}, -b^{\pm 1}$ can be identified as the endpoints of the cuts in the $Z = e^z$ plane with the constraint

$$\frac{1}{4}\Big(a + \frac{1}{a} + b + \frac{1}{b}\Big) = e^{t/2}. \tag{3.24}$$

The parameter ζ is related to the endpoints as

$$\zeta = \frac{1}{2}\Big(a + \frac{1}{a} - b - \frac{1}{b}\Big). \tag{3.25}$$

The eigenvalue density along the intervals may be computed from the discontinuity in the resolvent as was done in deriving (2.22) and integrated over each interval to obtain the interval's filling fraction (N_i/N) and hence the 't Hooft parameters t_i,

$$t_i = \frac{1}{4\pi\iota}\oint_{C_i} \omega_0(z)dz = \frac{1}{4\pi\iota}\oint_{C_i} \omega_0(Z)\frac{dZ}{Z}, \qquad i = 1, 2 \tag{3.26}$$

3.1.1 Perturbative Solution for Weak Coupling

The resolvent may be written as [9]

$$\frac{\omega_0(z)}{4} = \log\left(\frac{e^{-t/4}}{2}\left[\sqrt{(Z+b)(Z+1/b)} - \sqrt{(Z-a)(Z-1/a)}\right]\right) \tag{3.27}$$

At $t = 0$, the cuts shrink to points with $a = b = 1$. However, for small 't Hooft parameters, we may introduce two small parameters ϵ_1 and ϵ_2 such that

$$a + \frac{1}{a} = 2(1+\epsilon_1) \tag{3.28}$$

$$b + \frac{1}{b} = 2(1+\epsilon_2) \tag{3.29}$$

We wish to calculate ϵ_1 and ϵ_2 in terms of the 't Hooft parameters t_1 and t_2. To this end, we express the resolvent as

$$\frac{\omega_0(z)}{4} \sim \log\left(\sqrt{Z^2+2Z(1+\epsilon_2)+1} - \sqrt{Z^2-2Z(1+\epsilon_1)+1}\right) \tag{3.30}$$

in anticipation of using the formula (3.26), since the first term inside the logarithm was a holomorphic additive part and would not have contributed to the contour integral. To integrate the resolvent along C_1, we expand the resolvent around $\epsilon_2 = 0$ which gives,

$$\begin{aligned}
\omega_0(Z) \quad \sim \quad & 4\log\left((Z+1) - \sqrt{Z^2-2Z\epsilon_1-2Z+1}\right) \\
& + \frac{4Z}{(Z+1)\left((Z+1)-\sqrt{Z^2-2Z\epsilon_1-2Z+1}\right)}\,\epsilon_2 \\
& + \frac{2Z^2\left((Z+1)\sqrt{Z^2-2Z\epsilon_1-2Z+1}-2Z^2-4Z-2\right)}{(Z+1)^4\left((Z+1)-\sqrt{Z^2-2Z\epsilon_1-2Z+1}\right)^2}\,\epsilon_2^2 + O(\epsilon_2^3)
\end{aligned} \tag{3.31}$$

Note that one square root has vanished and this makes the computation tractable. As an illustration, we compute the coefficient to ϵ_2 in the the expression for the 't Hooft parameter t_1. The expression can be simplified as

$$\begin{aligned}
& \frac{4Z}{(Z+1)\left((Z+1)-\sqrt{Z^2-2Z\epsilon_1-2Z+1}\right)} \\
= \quad & \frac{4Z\left(\sqrt{Z^2-2Z\epsilon_1-2Z+1}+(Z+1)\right)}{2(Z+1)Z\,(\epsilon_1+2)} \\
= \quad & \frac{2\sqrt{Z^2-2Z\epsilon_1-2Z+1}}{(Z+1)\,(\epsilon_1+2)} + \frac{2}{\epsilon_1+2} \\
= \quad & \frac{\sqrt{Z^2-2Z\epsilon_1-2Z+1}}{(Z+1)\,(\mu(\epsilon_1))} + \frac{1}{\mu(\epsilon_1)}
\end{aligned} \tag{3.32}$$

where $\mu(\epsilon_1) = 1+\epsilon_1/2$. From (3.26) we see that the second term, being holomorphic in C_1, does not contribute and the first term gives a contribution equal to

$$\frac{\epsilon_2}{4\pi\iota\mu(\epsilon_1)} \oint_{C_1} \frac{\sqrt{Z^2-2Z\epsilon_1-2Z+1}}{(Z+1)} \frac{dZ}{Z} \equiv \frac{\epsilon_2}{4\pi\iota\mu(\epsilon_1)} \oint_{C_1} A(Z)dZ \tag{3.33}$$

From the residue theorem we have

$$\oint_{C_1} A(Z)dZ \quad = \quad -2\pi\iota\big(\mathrm{Res}(A,0)+\mathrm{Res}(A,-1)+\mathrm{Res}(A,\infty)\big) \tag{3.34}$$

$$= \quad -2\pi\iota(1-2\sqrt{\mu(\epsilon_1)}-1) = 4\pi\iota\sqrt{\mu(\epsilon_1)}. \tag{3.35}$$

Therefore, the contribution to t_1 at first order in ϵ_2 is

$$\frac{\epsilon_2}{\sqrt{\mu(\epsilon_1)}} \approx \epsilon_2\Big(1-\frac{\epsilon_1}{4}+\frac{3\epsilon_1^2}{32}-\frac{5\epsilon_1^3}{128}+\frac{35\epsilon_1^4}{2048}+O(\epsilon_1^5)\Big). \tag{3.36}$$

The expressions for 't Hooft parameters can be also inverted to give ϵ_1 and ϵ_2 in terms of t_1 and t_2.

3.2 ABJM Theory

The ABJM matrix model is given by [10]

$$Z_{ABJM}(N_1,N_2,g_s) \sim \int \prod_{i=1}^{N_1} \frac{d\mu_i}{2\pi} \prod_{j=1}^{N_2} \frac{d\nu_j}{2\pi} \Delta^2 e^{-\frac{1}{2g_s}(2\sum_i \mu_i^2 - 2\sum_j \nu_j^2)} \tag{3.37}$$

where

$$\Delta^2 = \frac{\prod_{i<j}\left(2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right)^2\left(2\sinh\left(\frac{\nu_i-\nu_j}{2}\right)\right)^2}{\prod_{i,j}\left(2\cosh\left(\frac{\mu_i-\nu_j}{2}\right)\right)^2}. \tag{3.38}$$

The free energy of the ABJM partition function is related to the lens space partition function by the analytic continuation $N_2 \to -N_2$. The natural 't Hooft parameters in the ABJM theory are given by

$$\lambda_j = \frac{N_j}{k} \tag{3.39}$$

where the Chern-Simons coupling k of ABJM theory is related to g_s by

$$g_s = \frac{4\pi\iota}{k}. \tag{3.40}$$

Since in the ABJM theory couplings $\lambda_{1,2}$ are real, matrix model couplings $t_{1,2}$ are purely imaginary.

The vacuum expectation values of the 1/6 and 1/2 BPS Wilson loops may be calculated from the matrix model using the relations [10]

$$\langle W_{\Box}^{1/6} \rangle = \oint_{C_1} \frac{dZ}{4\pi\iota}\omega(Z) \tag{3.41}$$

$$\langle W_{\Box}^{1/2} \rangle = \oint_{\infty} \frac{dZ}{4\pi\iota}\omega(Z). \tag{3.42}$$

As an example, we calculate the vacuum expectation value of the 1/2 BPS Wilson loop. We do the integral calculation for the Lens Space Model as we can later use the analytic continuation $t_2 \to -t_2$ to get the answer for the ABJM theory. Since the integrand in (3.42) is holomorphic in the annulus $A(0, R, \infty)$ for large R, as can be seen from (3.27),

$$\begin{aligned} \oint_{\infty} \frac{dZ}{4\pi\iota}\omega(Z) &= -\frac{2\pi\iota}{4\pi\iota}\,\mathrm{Res}(\omega, \infty) && (3.43) \\ &= (\epsilon_1 - \epsilon_2) = \zeta && (3.44) \end{aligned}$$

Note that out result is different from the one obtained in [10] as we have taken a different coupling to match with the potential used in the previous sections of this report and in [9].

Chapter 4

Conclusion and Further Work

In chapter 2 we used the saddle-point analysis and the resolvent technique to extract the eigenvalue distribution and free energy in the planar (genus 0) limit of large matrix dimension. Once the resolvent is obtained from the effective potential solving the Riemann-Hilbert, the eigenvalue distribution may be found with not much difficulty from a contour integral of the resolvent.

The techniques from chapter 2 easily generalise to two-matrix models, as we saw in chapter 3. For the lens space model we also derived the closed form solution of the planar resolvent. We then used this expression to calculate using perturbation the branch cuts of the planar resolvent in terms of the 't Hooft parameters. We showed how these relations can be used to solve quite conveniently for quantities such as expectation values of Wilson Loops which are contour integrals of the resolvent in the ABJM matrix model. We showed the calculations in detail for one order, the rest of the calculations proceed similarly.

During our study, we also browsed through free energy calculations with higher genus corrections, which are obtained from non-planar diagrams in the Feynman's diagrammatic approach and become significant for finite matrix dimension, and may provide a test for the AdS/CFT conjecture [11]. We also glanced over symmetric space classification of random matrix ensembles [13], and conformal field theory techniques used in the solving matrix models that also somewhat explain the universality of matrix models [12]. We hope to explore further such exciting properties and applications of random matrices and review them in the next stage of our report.

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