Topology and Geometry in Physics

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Contents

Abstract

The theory of topology has found many exciting applications in physical theories in the past century. It provides highly generalized methods of analyzing physical characteristics of a system, such as defects and singularities. In this report we lay out the basics of topology and differential geometry, which has a much larger history of applications in physics, illustrating facts with examples. We then present the analysis of Dirac's quantization rule for magnetic monopoles, which was very influential in demonstrating applications of topological methods in physics, at a time when they were remotely separate. We also see an example of topological defects in nematic liquid crystals.

Chapter 1

Introduction

This report is based on the study of initial chapters from the book *'Geometry, Topology and Physics'* by M. Nakahara [1] as a part of the undergraduate seminar.

In physics, many systems have such symmetries that allow us to identify groups of points as equivalent. In topology, we incorporate such symmetries in the structure of the space the system resides in. The new space is the quotient space under the made identifications. In *Chapter 2*, we describe an important theorem from group theory which allows us to identify the quotient spaces as the images of certain group mappings, and helps us better understand their structures [1].

In *Chapter 3*, we describe some of the general topological spaces and their properties [1]. We study the relations among the subspaces of a topological space under operations by the boundary operators. Although applications of such relations are not discussed in this report, they harmonize well with the differential analysis of space under the De Rham cohomology. We also see some properties of mappings from closed contours in one topological space to another. Such mappings are very useful in physics in understanding potentials in a closed region by only considering loops enclosing the region [2].

Two very exciting examples of such analysis are presented in *Chapter 5*, where we analyze the properties of systems due to singularities in the potentials (order parameters). Specifically we consider the Dirac monopole [1, 3, 4, 5, 6] where the singularity in the electro-magnetic potential is introduced upon assuming the existence of a magnetic monopole, and the line defects in the nematic liquid crystals [1, 7] where molecular orientations in the crystal characterize it into one of two classes with high transition energy.

When we consider such generalized topological spaces as choices for physical descriptions, we must be able to assign meanings to the important physical parameters such as positions and velocities. In *Chapter 4*, we develop the formal definitions of vector and tensor fields on smooth topological spaces, their transformations under change of basis, and the actions of derivative operators [1, 8].

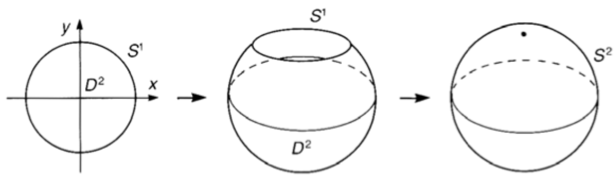
Chapter 2

Algebra

Let a map $\phi : G_1 \to G_2$ be a *group homomorphism*, then the kernel of the map is an equivalence class and the *quotient group* $G_1/ker(\phi)$ is isomorphic to the $im(\phi)$, where $G_1/ker(\phi)$ is the group of equivalence classes in which $a \circ n \sim a$ where $a \in G_1$ and $n \in ker(\phi)$.

Modular Arithmetic $\phi(5n + a) = a$ for $a \in \{0, 1, 2, 3, 4\}$ and $n \in \mathbb{Z}$, i.e. $\phi(n) =$ n (mod 5), is a homomorphism. The quotient group consists of the equivalence classes [a] such that $[a + 5n] = [a] + [5n] = [a]$ for , as $[5n] = [0]$.

Sphere If the boundary points S^1 of a disc D^2 are identified as equivalent, we get a sphere S^2. That is, $S^2 \cong D^2/S^1$.



Chapter 3

Topology

3.1 Homeomorphism

Homeomorphism is a continuous map from one topological space to another, with a continuous inverse. That is, two spaces are homeomorphic if they can be transformed continuously to one another without cutting or pasting. Homeomorphic spaces are topologically equivalent. Some of the topological invariants include connectedness, compactness and Euler characteristic which for a polygon is the (number of vertices) - (number of edges) + (number of faces).

Sphere The sphere S^2 is not homeomorphic to any open subset of $\mathbb{R}^2$ as the former is compact i.e. closed and bounded, while the latter is not. However, a sphere with a hole is homeomorphic to a disc D^2.

3.2 Homology

By $C_r(K)$, let us denote the free abelian group of all r-dimentional oriented volume elements of a topological space K such that $c = \sum_i c_i\sigma_i \in C_r(K)$ if $\sigma_i \in C_r(K)$, $c_i \in \mathbb{Z}$. For $r = 1$ it can be all curves, for $r = 2$ it can be all surfaces, and so on.
By $Z_r(K)$, we mean group of all elements of $C_r(K)$ that are r-cycles i.e. they do not have a boundary, which can happen if the boundary elements add up to 0. For example, closed loops. For a triangle the boundary is given by

$$\partial_1[(p_1p_2) + (p_2p_3) + (p_3p_1)] = (p_1 - p_2) + (p_2 - p_3) + (p_3 - p_1) = 0.$$

where p_i is a point and (p_ip_j) is an oriented line segment.
By $B_r(K)$ we mean group of all elements of $C_r(K)$ that are boundaries of $C_{r+1}(K)$.
It can be shown that $B_r(K) \subseteq Z_r(K)$.
Then the *homology group* is $H_r(K) \equiv Z_r(K)/B_r(K)$, which is the group of r-cycles

which are not themselves the boundaries of some element in C_{r+1}.

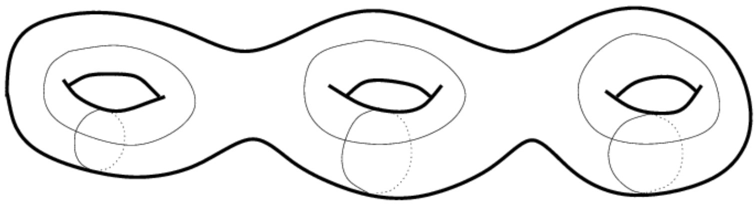
Circle Let $K = S^1$ i.e. a circle. $Z_1(K)$ is the free group generated by the closed loops which in this case only consist of the circle. Hence $Z_1(K) \cong \mathbb{Z}$. However, $C_2(K)$ is empty and hence so is $B_1(K)$. Therefore,

$$H_1(K) \cong Z_1(K)/B_1(K) \cong \mathbb{Z}/\{0\} \cong \mathbb{Z}.$$

One isomorphism map from H_1 to $\mathbb{Z}$ can be the *winding number*.

General surface Let K be the most general orientable two-dimensional surface consisting of a 2-sphere with h handles and q holes.
Each handle can be thought of as a torus (T^2) attached to the rest of the object. The first homology group of the torus is generated by two cycles i.e $H_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$. Each hole adds one cycle to the first homology group, which can be seen by taking a loop around the hole. However, adding one loop from around each hole gives a contractible loop and is ~ 0. Therefore,

$$H_1(K) \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{2h} \oplus \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{q-1}$$



Generators of first homology group of torus of genus 3

3.3 Homotopy

Two loops in a topological space are said to be homotopic if one can be continuously deformed into another. The product of two loops having a common point is the loop defined as: start from the common point, go around the first loop once and then the second loop once, reaching back to the starting point. Homotopy is an equivalence relation and the equivalence classes of loops hence formed in the topological space is the *fundamental homotopy group* of the space, denoted by $\pi_1(K)$. If the elements of this group commute, this group is isomorphic to the first homology group $H_1(K)$.

Chapter 4

Differential Geometry

4.1 Manifolds

It is not always possible to define a coordinate system on a topological space such that it is continuous everywhere and all close points have nearby coordinates. In such cases we use multiple coordinate *charts*, each defines coordinates locally on open subsets of space such that at points where more than one charts are defined, map from one to another should be smooth. A topological space together with set of charts covering the space, collectively called an *atlas*, is a *manifold*.

For example, if we use polar coordinates for sphere, we have a discontinuity as the point $(0, \pi/2)$ is very close to the point $(2\pi - \epsilon, \pi/2)$ for small ϵ, whereas, the coordinates are far. If we use stereographic coordinates from north pole, we get arbitrarily large coordinates for points close to north pole, they do not have close coordinate representations. Charts are *homeomorphisms* from open subsets of M to open subsets of $\mathbb{R}^m$, and being able to represent the sphere with a single chart would mean that sphere is homeomorphic to an open subset of $\mathbb{R}^2$, which is not true. Hence we need atleast two charts to cover a sphere. We will see an example of sphere manifold wen we study Dirac monopoles.

4.2 Vectors

A *smooth curve* in a space M is a smooth map $c : \mathbb{R} \to M$. Given a curve c and a function $f : M \to \mathbb{R}$, we can compose them and then differentiate to get

$$\frac{df(c(t))}{dt} = \frac{dx^\mu(c(t))}{dt}\frac{\partial f}{\partial x^\mu} = X^\mu \frac{\partial f}{\partial x^\mu} \equiv X[f]$$

where $x^\mu(p)$ gives the coordinate of point p in M and $X^\mu \equiv dx^\mu(c(t))/dt$. The map

$$T = X^\mu \frac{\partial}{\partial x^\mu} : f \to Tf = \frac{df}{dt}|_{t=0}, \qquad \text{where } c(0) = p$$

is a *tangent vector* at p with $\partial/\partial x^\mu$ as the coordinate basis. The space of tangent vectors at point p is the *tangent space* T_pM. Under change of coordinate system, the components will transform as

$$T = T^\mu \frac{\partial}{\partial x^\mu} = T^\nu \frac{\partial}{\partial x^\nu} \quad \Rightarrow \quad T^\nu = \frac{\partial x^\nu}{\partial x^\mu} T^\mu.$$

This transformation defines a *contravariant* vector.

The *differential 1-form* of a function, $f : M \to \mathbb{R}$, is a dual to the tangent vector defined by

$$< df, T > = \ Tf = df/dt \qquad \text{where } <,> \text{ is a bilinear map}$$

The space dual to tangent space T_pM is the *cotangent space* T^*_pM. dx^μ gives a basis for T^*_pM dual to the coordinate basis of vector

$$< dx^\mu, \frac{\partial}{\partial x^\mu} > = \ \delta^\mu_\nu$$

Under change of coordinate system, the components of the dual vector $\omega_\mu dx^\mu$ transform as a *covariant vector*

$$\omega_\nu = \omega_\mu \frac{\partial x_\mu}{\partial x_\nu}$$

A map $f : M \to N$ between two spaces induces a map f_* called the *differential map* $f_* : T_pM \to T_{f(p)}M$, such that

$$(f_*V)[g] \equiv V[gf]$$

Another induced map is the *pullback map* $f^* : T^*_{f(p)}N \to T^*_pM$ defined by

$$< f^*\omega, V > = < \omega, f_*V >$$

Mechanics In a configuration space Q, where q^i are local coordinates and $v^i = fq^i/dt = \dot{q}^i$ is the velocity of the dynamical trajectory q(t), a *Lagrangian* $L(q^i, v^j)$ is a function on the tangent bundle $L : TQ \to \mathbb{R}$. The tangent bundle is mapped to its cotangent bundle T^*Q by the *Legendre transformation* via

$$p_i = \frac{\partial L}{\partial v^i}$$

The fact that the momentum lies in the cotangent space can be seen from its covariant transformation under change of coordinate basis. If the above map is invertible, which can happen if the Lagrangian is a convex function of the v^i, we obtain an isomorphism between tangent and cotangent bundle, which is very similar to the *Wave-Particle Duality* where the particle velocity can be obtained from the wave vector. The Legendre transform of the Lagrangian with respect to fibre coordinates is the *Hamiltonian*

$$H(q^i, p_j) = p_i v^j(p_k) - L(q^i, v^j(p_k))$$

Since, H and L are invariant under coordinate basis change, we see another reason why momentum should be dual to the position.

4.3 Tensors

A *tensor* of type (q,r) is a multi-linear map : $\bigotimes^q T_p^* M \bigotimes^r T_p M \to \mathbb{R}$, written in terms of the basis as

$$\Omega = \Omega^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} \frac{\partial}{\partial x^{\mu_1}} \dots \frac{\partial}{\partial x^{\mu_q}} dx^{\nu_1} \dots dx^{\nu_r}$$

$$\Omega^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} = \Omega(dx^{\mu_1}, \dots, dx^{\mu_q}, \frac{\partial}{\partial x^{\nu_1}}, \dots, \frac{\partial}{\partial x^{\nu_r}})$$

A tensor of type (0,r) is a *covariant tensor* of rank r. A continuous assignment of an element of $\Im^q_{r,p} M$ at each point $p \in M$ is a *tensor field*, where $\Im^q_{r,p} M$ is the set of type (q,r) tensors at point p. If $(q, r) = (0, 1)$ it is a *co-vector field*, and if (1,0) it is a *vector field*.

4.4 Differential Forms

A *differential q-form* is an antisymmetric tensor of type (0,q). The vector space of q-forms acting on $\bigotimes^p T_p M$ is denoted by $\Lambda^q(T_p M)$. Dim $\Lambda^q(T_p M) = \binom{n}{q}$, where n is the dimensions of $T_p M$. $\Lambda^0(T_p M) = \mathbb{R}$ by convention.

The *exterior product* is an antisymmetric tensor product $\wedge : \Lambda^q(T_p M) \times \Lambda^r(T_p M) \to \Lambda^{q+r}(T_p M)$ such that

$$(\omega \wedge \xi)(V_1, \dots, V_{q+r}) = \frac{1}{q!r!} \sum_{P \in S_{q+r}} sgn(P) \omega(V_{P(1)}, \dots, V_{P(q)}) \xi(V_{P(q+1)}, \dots, V_{P(q+r)})$$

where $V_i \in T_pM$. The exterior product of r one-forms is

$$dx^{\mu_1} \wedge dx^{\mu_2} \cdots \wedge dx^{\mu_r} = \sum_{P \in S_r} sgn(P) dx^{\mu_1} \otimes dx^{\mu_2} \cdots \otimes dx^{\mu_r}$$

For example, $dx^\mu \wedge dx^\nu = dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu$.

The direst sum of vector spaces $\Lambda^q(T_pM)$, $\bigoplus_{q=0}^{n} \Lambda^q(T_pM)$, is denoted by $\Lambda^*(T_pM)$, and along with the exterior product forms an algebra.

The *exterior derivative* d is a map $\Lambda^r(T_pM) \to \Lambda^{r+1}(T_pM)$ whose action on an r-form

$$\omega = \frac{1}{r!} \omega_{\mu_1 \ldots \mu_r} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}$$

where $\omega_{\mu_1 \ldots \mu_r}$ is totally antisymmetric, is given by

$$d\omega = \frac{1}{r!} \left(\frac{\partial}{\partial x^\nu} \omega_{\mu_1 \ldots \mu_r} \right) dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}$$

The action of d^2 on ω is

$$d^2\omega = \frac{1}{r!} \left(\frac{\partial^2}{\partial x^\lambda \partial x^\nu} \omega_{\mu_1 \ldots \mu_r} \right) dx^\lambda \wedge dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r} = 0$$

as $\partial^2 \omega_{\mu_1 \ldots \mu_r} / \partial x^\lambda \partial x^\nu$ is symmetric with respect to λ and ν while $dx^\lambda \wedge dx^\nu$ is anti-symmetric.

Electromagnetic potential $A = (\phi, \mathbf{A})$ is a one-form, $A = A_\mu dx^\mu$. The *Electromagnetic tensor* F is defined by $F = dA$.

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = -E_i dt \wedge dx^i + \frac{1}{2} \epsilon_{ijk} B_k dx^i \wedge dx^j$$

where

$$\mathbf{E} = -\nabla\phi - \frac{\partial}{\partial x^0} \mathbf{A} \qquad \text{and} \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Using the identity $d^2 = 0$ we get two of the Maxwell's Equations,

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{and} \qquad \partial \mathbf{B} / \partial t = -\nabla \times \mathbf{E}$$

Chapter 5

Topological Defects

5.1 Gauge theory

From the identity $d^2\omega = 0$, we see that the potential $\tilde{A} = A + d\Lambda$ leads to the same electromagnetic tensor as A. Hence, we expect the trajectory of the particle to remain same under such *gauge transformation*. For the *time-independent electromagnetic fields*, $d\Lambda$ can be replaced by $\nabla\Lambda$.

The Hamiltonian for the system is given by

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{e\mathbf{A}}{c}\right)^2 + e\phi$$

and the *kinematical momentum* is given by

$$m\frac{dx_i}{dt} = m\frac{[x_i, H]}{\iota\hbar} = p_i - eA_i/c$$

Therefore, if the states change as $|\psi\rangle \to |\tilde{\psi}\rangle$ under gauge transformation, we expect

$$\begin{aligned}
\langle\psi|\mathbf{x}|\psi\rangle &= \langle\tilde{\psi}|\mathbf{x}|\tilde{\psi}\rangle \\
\langle\psi|\left(\mathbf{p} - \frac{e\mathbf{A}}{c}\right)|\psi\rangle &= \langle\tilde{\psi}|\left(\mathbf{p} - \frac{e\tilde{\mathbf{A}}}{c}\right)|\tilde{\psi}\rangle \\
\langle\psi|\psi\rangle &= \langle\tilde{\psi}|\tilde{\psi}\rangle
\end{aligned}$$

These conditions are satisfied by the transformation

$$|\tilde{\psi}\rangle = \exp\left[\frac{\iota e\Lambda(\mathbf{x})}{\hbar c}\right]|\psi\rangle$$

validity of which can be verified by comparing the original with the transformed Schrodinger equation.

5.2 Dirac Monopole

Although *magnetic monopoles* have not yet been found and Quantum Mechanics does not predict their existence, Dirac observed that if monopoles were to exist, it would lead to quantization of magnetic and electric charges.

Consider time-independent electromagnetic fields. In this case, the curvature tensor equals the magnetic field i.e.

$$F_{|\mathbb{R}^3} = B = \frac{1}{2} F_{ij} dx^i dx^j.$$

Dirac monopole introduces a singularity and hence, to describe it we use two coordinate patches $U_\pm$ to describe the $z > -\epsilon$ and $z < \epsilon$ regions of $\mathbb{R}^3 - \{0\}$ with overlap region $U_+ \cap U_-$ effectively equal to the x-y plane with z=0 minus the origin. We define the vector potentials in the two regions as

$$A_\pm = \frac{g}{r} \frac{1}{z \pm r}(xdy - ydx) = g(\pm 1 - \cos\theta) d\phi$$

The two potentials are related by the gauge transformation

$$A_+ = A_- + 2g \; d\tan^{-1}(y/x) = A_- + 2g \; d\phi$$

and hence give the same field. A_+ and A_- have the *Dirac string singularities* at $\theta = \pi$ and $\theta = 0$ respectively, however these strings lie outside the coordinate patches their respective potentials are defined in, and we get a genuine singularity only at $r = 0$.

Accordingly the gauge is given by

$$\Lambda = 2g\phi$$

which we see is singular at $\theta = 0, 2\pi$. But we use the transformation only at $\theta = \pi$ and hence the singularities do not show up in analysis.

The curvature of A is given by

$$F = g \sin\theta \; d\theta \wedge d\phi = \frac{g}{2r^3} \; \epsilon_{ijk} x^i \; dx^j \wedge dx^k$$

Hence the magnetic field is given by

$$B = \frac{g\mathbf{x}}{r^3}$$

and the flux is

$$\Phi = \oint_S \mathbf{B} \cdot d\mathbf{S} = 4\pi g$$

In analogy with electric charge, g can be identified as a measure of magnetic charge.

For this gauge, the wave-functions transform as

$$|\psi^+\rangle = \exp\left[\frac{\iota e\Lambda(\mathbf{x})}{\hbar c}\right]|\psi^-\rangle$$

We require the wave-function to be single valued as we go from $\phi = 0$ to $\phi = 2\pi$ along the equatior, which gives us the *Dirac quantization condition*

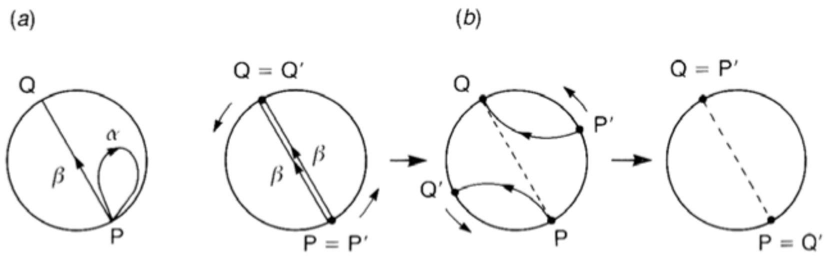
$$\frac{2eg}{\hbar c} = n$$

We see that at the equator, the transition function is of the form $e^{\iota n\phi}$, and hence maps the equator S^1 to U(1), the integer n characterizing the fundamental homotopy group $\pi_1(U(1)) = \mathbb{Z}$.

An interesting point in this analysis is that the quantization is not due to the discreteness of the spectrum of an operator in Hilbert space but rather due to topological considerations.

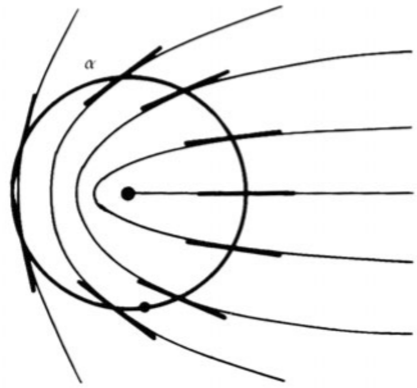
5.3 Defects in Nematic Liquid Crystals

The molecules of nematic liquid crystals are like rods with heads and tails. But they possess inversion symmetry and hence the orientation of a molecule can be described by a point on a sphere with antipodal points identified, i.e the *real projective space* $\mathbb{R}P^2$. The map $n(\mathbf{r}) : \mathbb{R}^3 \to \mathbb{R}P^2$, which describes the configuration at each point in the crystal space is called the *texture.*

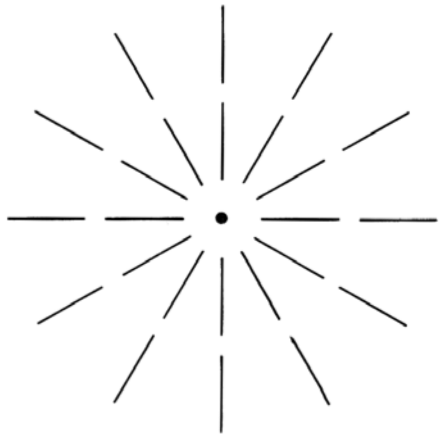


Loops in $\mathbb{R}P^2$. (a) α is a trivial loop while the loop β cannot be shrunk to a point. (b) $\beta \star \beta$ is continuously shrunk to a point.

Line Defects We see from the figure that $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2 = \{0, 1\}$. Therefore, there are two kinds of line defects in nematic liquid crystals, one can be continuously transformed into uniform configuration while the other cannot. More specifically, we take a loop γ around a line separated from the singular region by few molecular lengths so that the texture is well defined along the loop. The function $n(\mathbf{r})$ maps γ to some closed contour τ in $\mathbb{R}P^2$. τ may be of two types: (i) it starts and ends at the same point (for example, a circle) or, (ii) it connects the diametrically opposite points of S^2 which are equivalent in $\mathbb{R}P^2 = S^2/Z_2$. Contours of type (i) can be shrunk to a point, while contours of type (ii) cannot be, as shown in the part (a) of above figure. The latter represent stable vortices. They result from spontaneous symmetry breakdown and represent deviations from minimum energy configurations, contributing significantly to the gradient energy which makes their transformation into uniform configuration energetically almost impossible. In nematic liquid crystals it requires destruction of the nematic order in the whole half plane ending at the line. Two vortices of same type may however be deformed into one another with some expenditure of energy.



(a) Stable Vortex



(b) Fictitious Vortex

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