Chapter 3

Black Scholes

In these notes we price an option contract using the no-arbitrage pricing theory.

3.1 Black Scholes Option Pricing

Definition 3.1 (Black Scholes Model). *Black Scholes model is model for the market where the risk-free interest rate r is constant and tradable asset price process follows Geometric Brownian motion i,e.*

$$D_t \sim e^{-rt}$$
$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

We assume that the no-arbitrage condition holds in the market and that there is a risk neutral probability measure $\mathbb{P}$ such that discounted asset process is a local martingale under this measure.

$$\begin{aligned} d(D_tS_t)_{\text{drift}} &= 0 \\ \big(-rD_tS_tdt + D_tS_t(\mu dt + \sigma dW_t^{\mathbb{P}}) + 0\big)_{\text{drift}} &= 0 \\ (\mu - r)D_tS_t &= 0 \end{aligned}$$

Therefore we have $\mu = r$.
We know the solution to the Geometric Brownian motion

$$S_t = S_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t^{\mathbb{P}}}$$

Let's consider a call option on the stock S with expiration at time T and strike K. The value of this option at time T is

$$V_T = \max(S_T - K, 0)$$

From the no-arbitrage pricing theory we have D_tV_t is a local martinage and under bounded regularity conditions we have

$$D_tV_t = \mathbb{E}^{\mathbb{P}}[D_TV_T|\mathcal{F}_t] \tag{3.1}$$

i.e. D_tV_t is a martingale.

Solving we have,

$$
\begin{aligned}
V_t &= \frac{1}{D_t}\mathbb{E}^{\mathbb{P}}[D_T V_T|\mathcal{F}_t] \\
&= e^{rt}\mathbb{E}^{\mathbb{P}}[e^{-rT}\max(S_T - K, 0)|\mathcal{F}_t] \\
&= e^{-r(T-t)}\mathbb{E}^{\mathbb{P}}[\max(S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+(\sigma W_T^{\mathbb{P}}-\sigma W_t^{\mathbb{P}}} - K, 0)|\mathcal{F}_t] \\
&= e^{-r(T-t)}\int_{-\infty}^{\infty}\max(S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}x} - K, 0)\frac{1}{\sqrt{2\pi}}e^{-x^2}dx
\end{aligned}
$$

Denoting $\tau = T - t$ and $d_{1,2} = \frac{1}{\sigma\sqrt{\tau}}\big[\log(\frac{S_t}{K}) + (r \pm \frac{1}{2}\sigma^2)\tau\big]$, we have

$$
\begin{aligned}
V_t &= e^{-r\tau}\int_{-d_2}^{\infty}\big(S_t e^{(r-\frac{1}{2}\sigma^2)\tau+\sigma\sqrt{\tau}x} - S_t e^{(r-\frac{1}{2}\sigma^2)\tau+\sigma\sqrt{\tau}(-d_2)}\big)\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}dx \\
&= S_t e^{-\frac{1}{2}\sigma^2\tau}\int_{-d_2}^{\infty}\big(e^{\sigma\sqrt{\tau}x} - e^{\sigma\sqrt{\tau}(-d_2)}\big)\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}dx \\
&= S_t e^{-\frac{1}{2}\sigma^2\tau}\big[\int_{\infty}^{-d_2}e^{-\sigma\sqrt{\tau}x}\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}dx - \int_{-\infty}^{d_2}e^{\sigma\sqrt{\tau}(-d_2)}\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}dx\big] \\
&= S_t e^{-\frac{1}{2}\sigma^2\tau}\big[e^{\frac{1}{2}\sigma^2\tau}\int_{-d_2}^{\infty}\frac{1}{\sqrt{2\pi}}e^{\frac{-(x+\sigma\sqrt{\tau})^2}{2}}dx - e^{\sigma\sqrt{\tau}(-d_2)}\int_{-\infty}^{d_2}\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}dx\big] \\
&= S_t\int_{-d_2-\sigma\sqrt{\tau}}^{\infty}e^{-y^2}dy - Ke^{-r\tau}\int_{-\infty}^{d_2}\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}dx \\
&= S_t\int_{-\infty}^{d_2+\sigma\sqrt{\tau}}e^{-y^2}dy - Ke^{-r\tau}\int_{-\infty}^{d_2}\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}dx \\
&= S_t N(d_1) - Ke^{-r\tau}N(d_2)
\end{aligned}
$$

where N is the cumulative distribution function of the standard normal distribution.