Chapter 4

Numeraires

In these notes we introduce numeraires and theorems related to change of numeraires along with their applications.

4.1 Change of Measure

Definition 4.1 (Radon-Nikodym Derivative)**.** *Consider two equivalent probability measures $\mathbb{P}$ and $\hat{\mathbb{P}}$ on a measurable space (Ω, Σ). The Radon-Nikodym derivate $d\hat{\mathbb{P}}/d\mathbb{P} : \Omega \to \mathbb{R}$ is defined such that for any subset $A, \Omega \supseteqq A \in \Sigma$,*

$$\int_A d\hat{\mathbb{P}} = \int_A (d\hat{\mathbb{P}}/d\mathbb{P}) d\mathbb{P}. \tag{4.1}$$

Suppose $(\Omega, \mathcal{F}, \mathcal{F}_t)$ is a filtered probability space, then note from the above definition we have

$$(d\hat{\mathbb{P}}/d\mathbb{P})|_{\mathcal{F}_t} = \mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})|\mathcal{F}_t] \tag{4.2}$$

$(d\hat{\mathbb{P}}/d\mathbb{P})|_{\mathcal{F}_t}$ is also written as $(d\hat{\mathbb{P}}/d\mathbb{P})_t$ and is thus a martingale stochastic process (by iterated conditioning) in $\mathbb{P}$.

Example 4.1 Let's consider a simple example to better understand Radom Nikodym derivative. Consider a die roll. We can assign mulitple probability distributions to the outcomes.

ω	$\mathbb{P}$	$\hat{\mathbb{P}}$	$d\hat{\mathbb{P}}/d\mathbb{P}$
1	1/6	1/2	3
2	1/6	1/4	3/2
3	1/6	1/8	3/4
4	1/6	1/16	3/8
5	1/6	1/32	3/16
6	1/6	1/32	3/16

The probability in $\hat{\mathbb{P}}$ of getting an odd number is $\int_{\omega\in\{1,3,5\}} d\hat{\mathbb{P}} = 1/2 + 1/8 + 1/32 = 21/32 = 3 * 1/6 + 3/4 * 1/6 + 3/16 * 1/6 = \int_{\omega\in\{1,3,5\}} (d\hat{\mathbb{P}}/d\mathbb{P}) d\mathbb{P}$

Theorem 4.2 (Abstract Bayes' Theorem)**.** *Let $\mathbb{P}$ and $\hat{\mathbb{P}}$ be two measures on measurable space $(\Omega, \mathcal{F})$. Let $\mathcal{G} \subset \mathcal{F}$ be another sigma algebra on Ω. Then for any $A \in G$ and random variable X*

$$\mathbb{E}^{\hat{\mathbb{P}}}[X|G] = \frac{\mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})X|G]}{\mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})|G]}. \tag{4.3}$$

Proof. We show that for any $A \in G$

$$\mathbb{E}^{\hat{\mathbb{P}}}[X|G]\mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})|G] = \mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})X|G]$$

Since the random variables involved are constant over A, we can check equality on integrals over A.

$$\begin{aligned}
\int_A \mathbb{E}^{\hat{\mathbb{P}}}[X|G]\mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})|G]d\mathbb{P} &= \int_A \mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})\mathbb{E}^{\hat{\mathbb{P}}}[X|G]|G]d\mathbb{P} && (\mathbb{E}^{\hat{\mathbb{P}}}[X|G] \text{ is G measurable}) \\
&= \int_A (d\hat{\mathbb{P}}/d\mathbb{P})\mathbb{E}^{\hat{\mathbb{P}}}[X|G]d\mathbb{P} && (\text{definition of conditional expectation}) \\
&= \int_A \mathbb{E}^{\hat{\mathbb{P}}}[X|G]d\hat{\mathbb{P}} && (\text{definition of Radon Nikodym derivative}) \\
&= \int_A X d\hat{\mathbb{P}} && (\text{definition of conditional expectation}) \\
&= \int_A (d\hat{\mathbb{P}}/d\mathbb{P})X d\mathbb{P} && (\text{definition of Radon Nikodym derivative}) \\
&= \int_A \mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})X|G]d\mathbb{P} && (\text{definition of conditional expectation}).
\end{aligned}$$

□

Remark. *Taking $X = V_T$ where V_s is $\mathcal{F}_s$ adapted and $G = \mathcal{F}_t$ for $t < T$ in the above theorem, we get the very useful formula for measure change for conditional expectations on filtered spaces,*

$$\mathbb{E}^{\hat{\mathbb{P}}}[V_T|\mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}[\frac{(d\hat{\mathbb{P}}/d\mathbb{P})_T}{(d\hat{\mathbb{P}}/d\mathbb{P})_t}V_T|\mathcal{F}_t] \tag{4.4}$$

Proof.

$$\begin{aligned}
\mathbb{E}^{\hat{\mathbb{P}}}[V_T|\mathcal{F}_t] &= \frac{\mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})V_T|\mathcal{F}_t]}{\mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})|\mathcal{F}_t]} && (\text{abstract Bayes' theorem}) \\
&= \frac{\mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})V_T|\mathcal{F}_T]|\mathcal{F}_t]}{\mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})|\mathcal{F}_t]} && (\text{iterated conditioning}) \\
&= \frac{\mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})|\mathcal{F}_T]V_T|\mathcal{F}_t]}{\mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})|\mathcal{F}_t]} && (V_T \text{ is } \mathcal{F}_T \text{ measureable}) \\
&= \frac{\mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})_T V_T|\mathcal{F}_t]}{(d\hat{\mathbb{P}}/d\mathbb{P})_t} && (\text{martigale property of Radon Nikodym derivative}) \\
&= \mathbb{E}^{\mathbb{P}}[\frac{(d\hat{\mathbb{P}}/d\mathbb{P})_T}{(d\hat{\mathbb{P}}/d\mathbb{P})_t}V_T|\mathcal{F}_t] && ((d\hat{\mathbb{P}}/d\mathbb{P})_t \text{ is } \mathcal{F}_t \text{ measurable}).
\end{aligned}$$

□

We consider processes until terminal time S i.e. $\mathcal{F} = \mathcal{F}_S$ and $(d\hat{\mathbb{P}}/\mathbb{P}) = (d\hat{\mathbb{P}}/\mathbb{P})_S$. We take a strictly positive martingale process to be the Random Nikodym derivative:

$$df_t = f_t\sigma(t)dW_t;\;\; f_t = e^{-\int_0^t \frac{1}{2}\sigma^2(s)ds + \int_0^t \sigma(s)dW_s}. \tag{4.5}$$

Theorem 4.3 (Girsanov Theorem)**.** *If $W_{1,t}$ is a Brownian motion in $\mathbb{P}$ and the Radon-Nikodym derivative $(d\hat{\mathbb{P}}/d\mathbb{P})_t = f_t$ is given by $f_t = e^{-\int_0^t \frac{1}{2}\sigma^2(s)ds + \int_0^t \sigma(s)dW^{\mathbb{P}}_{2,s}}$ then $W_{1,t} - \int_0^t \sigma(s)dW_{1,s}dW_{2,s}$ is a Brownian motion in $\hat{\mathbb{P}}$.*

Proof. We show that $X_t = W_{1,t} - \int_0^t \sigma(s)dW_{1,s}dW_{2,s}$ follows normal distribution with mean 0 and variance t in $\hat{\mathbb{P}}$. Other required properties can be verified easily. We show that the moment generating function of X_t is same as that of normal distribution with mean 0 and variance t.

$$\begin{aligned}
\mathbb{E}^{\hat{\mathbb{P}}}[e^{-yX_t}] &= \mathbb{E}^{\mathbb{P}}[(d\hat{\mathbb{P}}/d\mathbb{P})_t e^{-yX_t}] \\
&= \mathbb{E}^{\mathbb{P}}[e^{-\int_0^t \frac{1}{2}\sigma^2(s)ds + \int_0^t \sigma(s)dW_{2,s}} e^{-yX_t}] \\
&= e^{-\int_0^t \frac{1}{2}\sigma^2(s)ds}\mathbb{E}^{\mathbb{P}}[e^{\int_0^t \sigma(s)dW_{2,s} - yX_t}] \\
&= e^{-\int_0^t \frac{1}{2}\sigma^2(s)ds}\mathbb{E}^{\mathbb{P}}[e^{\int_0^t \sigma(s)dW_{2,s} - y(W_{1,t} - \int_0^t \sigma(s)dW_{1,s}dW_{2,s})}] \\
&= e^{-\int_0^t \frac{1}{2}\sigma^2(s)ds}\mathbb{E}^{\mathbb{P}}[e^{\int_0^t (\sigma(s)dW_{2,s} - ydW_{1,s} + y\sigma(s)dW_{1,s}dW_{2,s})}].
\end{aligned}$$

Define the martingale $Z_{y,t} = \int_0^t (\sigma(s)dW_{2,s} - ydW_{1,s})$. Then $e^{Z_{y,t} - \frac{1}{2}\int_0^t dZ_{y,s}dZ_{y,s}}$ is a martingale as well, with

$$dZ_{y,s}dZ_{y,s} = (\sigma^2(s) + y^2)ds - 2y\sigma dW_{1,s}dW_{2,s}.$$

We then have

$$\begin{aligned}
\mathbb{E}^{\hat{\mathbb{P}}}[e^{-yX_t}] &= e^{-\int_0^t \frac{1}{2}\sigma^2(s)ds}\mathbb{E}^{\mathbb{P}}[e^{Z_{y,t} - \frac{1}{2}\int_0^t dZ_{y,s}dZ_{y,s} + \frac{1}{2}\int_0^t (\sigma^2(s)+y^2)ds}] \\
&= e^{\frac{1}{2}y^2 t}\mathbb{E}^{\mathbb{P}}[e^{Z_{y,t} - \frac{1}{2}\int_0^t dZ_{y,s}dZ_{y,s}}] \\
&= e^{\frac{1}{2}y^2 t}(e^{Z_{y,t} - \frac{1}{2}\int_0^t dZ_{y,s}dZ_{y,s}})|_{t=0} \\
&= e^{\frac{1}{2}y^2 t}.
\end{aligned}$$

□

4.2 Numeraires

Definition 4.4 (Numeraire)**.** *A Numeraire is a strictly positive price process of a tradable relative to which prices of all other tradables are expressed.*

A savings account which earns interest at the instantaneous interest rate can be taken as a numeraire. The value of the savings account at any point is given by

$$A_t = e^{\int_0^t r(t)dt} = 1/D(t) \tag{4.6}$$

where $D(t)$ is the discount factor.

The fact that discounted trade prices are martingales in risk-neutral measure can then also be stated as: tradable prices in savings account (or money market) numeraire are martingales.

Consider another numeraire N_t. Since the numeraire is itself a price process, $D_t N_t$ is a martingale and we can take it as a Radon-Nikodym derivative, with a suitable normalization so that $\mathbb{E}^{\mathbb{P}}[(d\mathbb{N}/d\mathbb{P})] = 1$, giving

$$\mathbb{E}^{\mathbb{N}}[\frac{X_T}{N_T}|\mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}[\frac{(d\mathbb{N}/d\mathbb{P})_T}{(d\mathbb{N}/d\mathbb{P})_t}\frac{X_T}{N_T}|\mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}[\frac{D_T N_T}{D_t N_t}\frac{X_T}{N_T}|\mathcal{F}_t] = \frac{1}{D_t N_t}\mathbb{E}^{\mathbb{P}}[D_T X_T|\mathcal{F}_t] = \frac{X_t}{N_t} \tag{4.7}$$

where $\mathbb{P}$ is the risk neutral measure and $\mathbb{N}$ is the measure corresponding to the numeraire N_t.

Remark. *The above equation shows that tradable prices with respect to a numeraire are martingales in the measure corresponding to the numeraire.*

Risk neutral measure corresponds to the savings account (or money market) numeraire.

Definition 4.5 (T-forward measure)**.** *If we take N_t to be the price of a riskless bond maturing at time T, the corresponding measure is the T-forward measure.*

$$X_t/B(t,T) = \mathbb{E}^T[X_T/B(T,T)] = \mathbb{E}^T[X_T] \tag{4.8}$$

We see that that expectation of X_T in the T-forward measure gives the forward price $X_t/B(t,T)$ of the trade with expiration date T, hence the name T-forward measure. From the above equation we also notice that expiration T forward price process on an asset is martingale in T-forward measure.

Example 4.2 Option Pricing To see how measure changes can be used in pricing, let's take the example of Black Scholes option pricing. For an option on stock with expiry at T and strike K, we have the valuation forumula

$$V_t = \frac{1}{D_t}\mathbb{E}^{\mathbb{P}}[D_T \max(S_T - K, 0)|\mathcal{F}_t]$$

where $\mathbb{P}$ is the risk neutral probability measure, D_t is the discount factor and S_t is the stock price which follows the Black Scholes diffusion

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t$$

where r is the riskfree interest rate and W_t is a Brownian motion in $\mathbb{P}$.

Denote by $\mathbb{I}_{S_T>K}$ the indicator variable which takes value 1 is $S_T > K$ and 0 otherwise. We then have

$$\begin{aligned} V_t &= \frac{1}{D_t}\mathbb{E}^{\mathbb{P}}[D_T \mathbb{I}_{S_T>K}(S_T - K)|\mathcal{F}_t] \\ &= \frac{1}{D_t}\mathbb{E}^{\mathbb{P}}[D_T \mathbb{I}_{S_T>K} S_T|\mathcal{F}_t] - \frac{1}{D_t}\mathbb{E}^{\mathbb{P}}[D_T \mathbb{I}_{S_T>K} K|\mathcal{F}_t] \end{aligned}$$

Taking $\mathbb{S}$ to be the measure corresponding to stock price as numeraire, and taking $\mathbb{T}$ to be the measure corresponding to T-expiry bond as numeraire, we have

$$\begin{aligned}
V_t &= \frac{1}{D_t}\mathbb{E}^{\mathbb{S}}[D_T\frac{(d\mathbb{P}/d\mathbb{S})_T}{(d\mathbb{P}/d\mathbb{S})_t}\mathbb{I}_{S_T>K}S_T|\mathcal{F}_t] - \frac{1}{D_t}\mathbb{E}^{\mathbb{T}}[D_T\frac{(d\mathbb{P}/d\mathbb{T})_T}{(d\mathbb{P}/d\mathbb{T})_t}\mathbb{I}_{S_T>K}K|\mathcal{F}_t] \\
&= \frac{1}{D_t}\mathbb{E}^{\mathbb{S}}[D_T\frac{(d\mathbb{S}/d\mathbb{P})_t}{(d\mathbb{S}/d\mathbb{P})_T}\mathbb{I}_{S_T>K}S_T|\mathcal{F}_t] - \frac{1}{D_t}\mathbb{E}^{\mathbb{T}}[D_T\frac{(d\mathbb{T}/d\mathbb{P})_t}{(d\mathbb{T}/d\mathbb{P})_T}\mathbb{I}_{S_T>K}K|\mathcal{F}_t] \\
&= \frac{1}{D_t}\mathbb{E}^{\mathbb{S}}[D_T\frac{D_tS_t}{D_TS_T}\mathbb{I}_{S_T>K}S_T|\mathcal{F}_t] - \frac{1}{D_t}\mathbb{E}^{\mathbb{T}}[D_T\frac{D_t(B(t,T))}{D_TB(T,T)}\mathbb{I}_{S_T>K}K|\mathcal{F}_t] \\
&= S_t\mathbb{E}^{\mathbb{S}}[\mathbb{I}_{S_T>K}|\mathcal{F}_t] - KB(t,T)\mathbb{E}^{\mathbb{T}}[\mathbb{I}_{S_T>K}|\mathcal{F}_t]
\end{aligned}$$

where $B(t,T)$ is the bond price with the diffusion

$$dB(t,T)/B(t,T) = rdt + \sigma_2 dW_{2,t}$$

where $W_{2,t}$ is another Brownian motion in $\mathbb{P}$ such that $dW_t dW_{2,t} = \rho dt$.

Using Girsaonov's theorem, we have

$$\begin{aligned}
dW_t^{\mathbb{S}} &= dW_t^{\mathbb{P}} - \sigma dW_t^{\mathbb{P}} dW_t^{\mathbb{P}} &&= dW_t^{\mathbb{P}} - \sigma dt \\
dW_t^{\mathbb{T}} &= dW_t^{\mathbb{P}} - \sigma_2 dW_t^{\mathbb{P}} dW_{2,t}^{\mathbb{P}} &&= dW_t^{\mathbb{P}} - \sigma_2\rho dt
\end{aligned}$$

as Brownian motions in the measures corresponding to S, and B(t,T) numeraires respectively.

$$\frac{dS_t}{S_t} = rdt + \sigma(dW_t^{\mathbb{S}} + \sigma dt) = rdt + \sigma(dW_t^{\mathbb{T}} + \sigma_2\rho dt) \tag{4.9}$$

where $dW_t^{\mathbb{S}} = dW_t - \sigma dt$ is a Brownian motion in $\mathbb{S}$ and $dW_t^{\mathbb{T}} = dW_t^{\mathbb{S}} - \sigma_2\rho dt$ is a Brownian motion in $\mathbb{T}$. Rearranging,

$$\frac{dS_t}{S_t} = (r+\sigma^2)dt + \sigma dW_t^{\mathbb{S}} = (r+\sigma\sigma_2\rho)dt + \sigma dW_t^{\mathbb{T}} \tag{4.10}$$

$$S_T = S_t e^{(r+\frac{1}{2}\sigma^2)\tau + \sigma W_\tau^{\mathbb{S}}} = S_t e^{(r+\sigma(\sigma_2\rho - \frac{1}{2}\sigma)\tau + \sigma W_\tau^{\mathbb{T}}} \tag{4.11}$$

where $\tau = T - t$.

$$\begin{aligned}
\mathbb{E}^{\mathbb{S}}[\mathbb{I}_{S_T>K}|\mathcal{F}_t] &= \mathbb{E}^{\mathbb{S}}[\mathbb{I}_{S_t e^{(r+\frac{1}{2}\sigma^2)\tau+\sigma W_\tau^{\mathbb{S}}}>K}|\mathcal{F}_t] \\
&= \mathbb{E}^{\mathbb{S}}[\mathbb{I}_{(r+\frac{1}{2}\sigma^2)\tau+\sigma W_\tau^{\mathbb{S}}>\log(K/S_t)}|\mathcal{F}_t] \\
&= \mathbb{E}^{\mathbb{S}}[\mathbb{I}_{\sigma W_\tau^{\mathbb{S}}>\log(K/S_t)-(r+\frac{1}{2}\sigma^2)\tau}|\mathcal{F}_t] \\
&= \mathbb{E}^{\mathbb{S}}[\mathbb{I}_{\frac{1}{\sqrt{\tau}}W_\tau^{\mathbb{S}}>\frac{1}{\sigma\sqrt{\tau}}(\log(K/S_t)-(r+\frac{1}{2}\sigma^2)\tau)})|\mathcal{F}_t] \\
&= N(-\frac{1}{\sigma\sqrt{\tau}}(\log(K/S_t) - (r+\frac{1}{2}\sigma^2)\tau)) \\
&= N(d_1)
\end{aligned}$$

where N is the cumulative normal distribution, and $d_1 = \frac{1}{\sigma\sqrt{\tau}}(\log(S_t/K) + (r+\frac{1}{2}\sigma^2)\tau)$.

Similarly we have,

$$\begin{aligned}
\mathbb{E}^{\mathbb{T}}[\mathbb{I}_{S_T>K}|\mathcal{F}_t] &= \mathbb{E}^{\mathbb{T}}[\mathbb{I}_{S_t e^{(r+\sigma(\sigma_2\rho-\frac{1}{2}\sigma)\tau+\sigma W^{\mathbb{S}}_{\tau}}>K}|\mathcal{F}_t] \\
&= \mathbb{E}^{\mathbb{T}}[\mathbb{I}_{(r+\sigma(\sigma_2\rho-\frac{1}{2}\sigma)\tau+\sigma W^{\mathbb{S}}_{\tau}>\log(K/S_t)}|\mathcal{F}_t] \\
&= \mathbb{E}^{\mathbb{T}}[\mathbb{I}_{\sigma W^{\mathbb{S}}_{\tau}>\log(K/S_t)-(r+\sigma(\sigma_2\rho-\frac{1}{2}\sigma)\tau}|\mathcal{F}_t] \\
&= \mathbb{E}^{\mathbb{T}}[\mathbb{I}_{\frac{1}{\sqrt{\tau}}W^{\mathbb{S}}_{\tau}>\frac{1}{\sigma\sqrt{\tau}}(\log(K/S_t)-(r+\sigma(\sigma_2\rho-\frac{1}{2}\sigma)\tau)}|\mathcal{F}_t] \\
&= N(-\frac{1}{\sigma\sqrt{\tau}}(\log(K/S_t)-(r+\sigma(\sigma_2\rho-\frac{1}{2}\sigma)\tau)) \\
&= N(d_2)
\end{aligned}$$

where N is the cumulative normal distribution, and $d_2 = \frac{1}{\sigma\sqrt{\tau}}(\log(S_t/K)+(r+\sigma\sigma_2\rho-\frac{1}{2}\sigma^2)\tau)$.

And finally we have,

$$\begin{aligned}
V_t &= S_t N(d_1) - KB(t,T)N(d_2) \\
&= S_t N(d+\sigma\sqrt{\tau}) - KB(t,T)N(d+\sigma_2\rho\sqrt{\tau})
\end{aligned}$$

where $d = \frac{1}{\sigma\sqrt{\tau}}(\log(S_t/K)+(r-\frac{1}{2}\sigma^2)\tau)$.

Note that $\sigma d\tau$ and $\sigma_2\rho d\tau$ are the drift correction terms we get to stock price Brownian motion using Girsanov's theorem to change measure to stock and bond numeraires respectively.